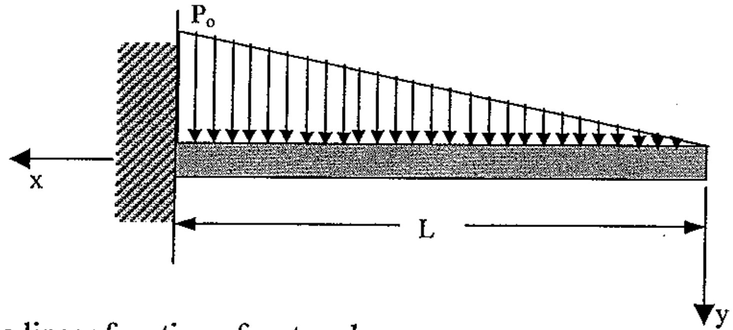
Problem 1 (Referred to Problem 5.10 in the 1st text, P232)

For a thin cantilever of Fig. P5.10, determine the stress function is given as below and find the corresponding stresses.

$$\phi = -c_1 xy + c_2 \frac{x^3}{6} - c_3 \frac{x^3 y}{6} - c_4 \frac{xy^3}{6} - c_5 \frac{x^3 y^3}{9} - c_6 \frac{xy^5}{20}$$



Hint: σ_y is a linear function of x at y=-h

Soln: Select the Airy Stress Function with the corresponding σ_y that contains zero and linear function of x terms.

ϕ_6 given below

	σ_x	σ_y	τ_{xy}	
$\frac{1}{30} a_6 x^6$	0	$a_6 x^4$	0	$\nabla^4 \phi = 0$ gives
$\frac{1}{20} b_6 x^5 y$	0	$b_6 x^4 y$	$-\frac{1}{4} b_6 x^4$	$e_6 = -(6a_6 + 2c_6)$
$\frac{1}{12} c_6 x^4 y^2$	$c_6 x^4$	$c_6 x^2 y^2$	$-\frac{2}{3} c_6 x^3 y$	$f_6 = -(b_6 + 2d_6)$
$\frac{1}{6} d_6 x^3 y^3$	$d_6 x^3 y$	$d_6 x y^3$	$-\frac{3}{2} d_6 x^2 y^2$	$c_6 = -(2e_6 + 6g_6)$
$\frac{1}{12} e_6 x^2 y^4$	$e_6 x^2 y^2$	$e_6 y^4$	$-\frac{2}{3} e_6 x y^3$	
$\frac{1}{20} f_6 x y^5$	$f_6 x y^3$	0	$-\frac{1}{4} f_6 y^4$	
$\frac{1}{30} g_6 y^6$	$g_6 y^4$	0	0	

ϕ is selected as

$$\phi = b_2 xy + \frac{1}{2}c_2 y^2 + \frac{1}{6}a_3 x^3 + \frac{c_3}{2}xy^2 + \frac{d_3}{6}y^3 + \frac{b_4}{6}x^3 y + \frac{d_4}{6}xy^3 +$$
$$\frac{e_4}{12}y^4 + \frac{c_5}{6}x^3y^2 + \frac{e_5}{12}xy^4 + \frac{f_5}{20}y^5 + \frac{d_6}{6}x^3y^3 + \frac{f_6}{20}xy^5 + \frac{g_6}{30}y^6$$

$\nabla^4\phi = 0$ gives

$$b_5 = -(2d_5 + f_5) \qquad d_5 = b_5 = 0 \Rightarrow f_5 = 0 \qquad (1)$$

$$e_5 = -(3a_5 + 2c_5) \qquad a_5 = 0 \Rightarrow e_5 = -2c_5 \qquad (2)$$

$$e_4 = -(a_4 + 2c_4) \qquad a_4 = c_4 = 0 \Rightarrow e_4 = 0 \qquad (3)$$

$$c_6 = -(2e_6 + 6g_6) \qquad e_6 = 0 = c_6 \Rightarrow g_6 = 0 \qquad (4)$$

$$f_6 = (-b_6 - 2d_6) \qquad b_6 = 0 \Rightarrow f_6 = -2d_6 \qquad (5)$$

The corresponding stress components are

$$\left.\begin{aligned}
\sigma_x &= c_2 + e_3 x + d_3 y + d_4 xy + \frac{1}{3}c_5 x^3 + e_5 xy^2 + d_6 x^3 y + f_6 xy^3 \\
\sigma_y &= a_3 x + b_4 xy + c_5 xy^2 + d_6 xy^3 \\
\tau_{xy} &= -b_2 - c_3 y - \frac{1}{2}b_4 x^2 - \frac{1}{2}d_4 y^2 - c_5 x^2 y - \frac{1}{3}e_5 y^3 - \frac{3}{2}d_6 x^2 y^2 - \frac{1}{4}f_6 y^4
\end{aligned}\right\} (6)$$

<u>Boundary Conditions</u>

$$\sigma_x(0, y) = 0 \Rightarrow c_2 + d_3 y = 0 \Rightarrow c_2 = d_3 = 0 \qquad (7)$$

$$\sigma_y(x, \frac{h}{2}) = 0 \qquad a_3 + \frac{h}{2}b_4 + \frac{h^2}{4}c_5 + \frac{h^3}{8}d_6 = 0 \tag{8}$$

$$\sigma_y(x, -\frac{h}{2}) = -\frac{P_0 x}{Lt} \qquad a_3 - \frac{h}{2}b_4 + \frac{h^2}{4}c_5 - \frac{h^3}{8}d_6 = -\frac{P_0}{Lt} \tag{9}$$

(8) + (9) $$a_3 + \frac{h^2}{4}c_5 = -\frac{P_0}{2Lt} \tag{10}$$

$$\tau_{xy}(x, \frac{h}{2}) = 0$$

$$-b_2 - \frac{h}{2}c_3 - \frac{b_4}{2}x^2 - \frac{h^2}{8}d_4 - \frac{h}{2}c_5 x^2 - \frac{h^3}{24}e_5 - \frac{3h^2}{8}d_6 x^2 - \frac{f_6}{64}h^4 = 0$$

$$b_2 + \frac{h}{2}c_3 + \frac{h^2}{8}d_4 + \frac{h^3}{24}e_5 + \frac{f_6}{64}h^4 = 0 \tag{11}$$

$$\frac{b^4}{2} + \frac{h}{2}c_5 + \frac{3}{8}h^2 d_6 = 0 \tag{12}$$

$\tau_{xy}(x, -\frac{h}{2}) = 0$ gives

$$-b_2 + \frac{h}{2}c_3 - \frac{h^2}{8}d_4 + \frac{h^3}{24}e_5 - \frac{f_6}{64}h^4 = 0 \tag{13}$$

$$-\frac{b^4}{2} + \frac{c_5}{2}h - \frac{3}{8}h^2 d_6 = 0 \tag{14}$$

(12) + (14) gives $c_5 = 0$ (15)

Eqn. (2) gives $e_5 = 0$ (16)

(11) + (13) gives $c_3 = 0$ (17)

Substituting (15) into (10) & (14), we obtain

$$a_3 = -\frac{P_o}{2Lt} \tag{18}$$

$$b_4 = -\frac{3}{4}h^2 d_6 \tag{19}$$

Rewriting eqn(13),

$$b_2 + \frac{h^2}{8}d_4 + \frac{h^4}{64}f_6 = 0 \tag{20}$$

Substituting (18), (19) & (15), into (8), we have

$$-\frac{P_o}{2Lt} + \frac{h}{2}\left(-\frac{3}{4}h^2 d_6\right) + 0 + \frac{h^3}{8}d_6 = 0$$

$$d_6 = -\frac{2P_o}{h^3 L t} \tag{21}$$

Eqn (5) gives $\quad f_6 = -2d_6 = \dfrac{4P_o}{h^3 L t}$ (22)

Eqn (19) gives $\quad b_4 = \dfrac{3}{2}\dfrac{P_o}{hLt}$ (23)

<u>B.C.</u>

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy}\Big|_{x=0} t\,dy = 0$$

$$\tau_{xy} = -b_2 - \frac{b_4}{2}x^2 - \frac{1}{2}d_4 y^2 - \frac{3}{2}d_6 x^2 y^2 - \frac{1}{4}f_6 y^4$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(-b_2 - \frac{1}{2}d_4 y^2 - \frac{1}{4}f_6 y^4\right) dy = 0$$

$$b_2 + \frac{h^2}{24} d_4 + \frac{h^4}{320} f_6 = 0 \qquad (24)$$

$$b_2 + \frac{h^2}{8} d_4 + \frac{h^4}{64} f_6 = 0 \qquad (20)$$

(20) - (24) gives

$$d_4 = -\frac{3}{20} h^2 f_6 \overset{\text{Eqn (22)}}{=} -\frac{3}{5}\frac{P_0}{hLt} \qquad (25)$$

Eqn.(20)

$$b_2 = -\frac{h^2}{8}\left(-\frac{3}{5}\frac{P_0}{hLt}\right) - \frac{h^4}{64}\left(\frac{4P_0}{hLt}\right)$$

$$b_2 = \frac{1}{80}\frac{P_0 h}{Lt} \qquad (26)$$

Summary

$b_2 = \frac{1}{80}\frac{P_0 h}{Lt}$, $c_2 = 0$ $\quad q_3 = -\frac{P_0}{2Lt}$

$c_3 = 0$ $\quad d_3 = 0$

$b_4 = \frac{3}{2}\frac{P_0}{hLt}$ $\quad d_4 = -\frac{3}{5}\frac{P_0}{hLt}$ $\quad e_4 = 0$

$c_5 = 0$ $\quad e_5 = 0$ $\quad f_5 = 0$

$d_6 = -\frac{2P_0}{h^3 Lt}$ $\quad f_6 = \frac{4P_0}{h^3 Lt}$ $\quad g_6 = 0$

$$\Phi = \frac{1}{80}\frac{P_o h}{Lt}xy - \frac{P_o}{12Lt}x^3 + \frac{1}{4}\frac{P_o}{hLt}x^3y - \frac{1}{10}\frac{P_o}{hLt}xy^3$$

$$-\frac{1}{3}\frac{P_o}{h^3Lt}x^3y^3 + \frac{1}{5}\frac{P_o}{h^3Lt}xy^5$$

Comparing with

$$\Phi = -C_1 xy + C_2\frac{x^3}{6} - C_3\frac{x^3y}{6} - C_4\frac{xy^3}{6} - C_5\frac{x^3y^3}{9} - C_6\frac{xy^5}{20}$$

$$C_1 = -\frac{1}{80}\frac{P_o h}{Lt}$$

$$C_2 = -\frac{1}{2}\frac{P_o}{Lt}$$

$$C_3 = -\frac{3}{2}\frac{P_o}{hLt}$$

$$C_4 = \frac{3}{5}\frac{P_o}{hLt}$$

$$C_5 = \frac{3P_o}{h^3Lt}$$

$$C_6 = -4\cdot\frac{P_o}{h^3Lt}$$

The corresponding stress components are

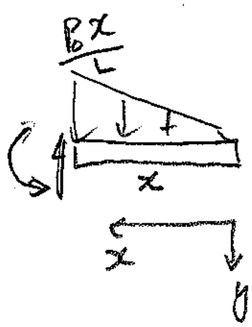
$$\sigma_x = -\frac{3}{5}\frac{P_o}{hLt}xy - \underline{\frac{2P_o}{h^3Lt}x^3y} + \frac{4P_o}{h^3Lt}xy^3$$

$$\sigma_y = -\frac{P_o}{2Lt}x + \frac{3}{2}\frac{P_o}{hLt}xy - \frac{2P_o}{h^3Lt}xy^3$$

$$\tau_{xy} = -\frac{1}{80}\frac{P_o h}{Lt} - \frac{3}{4}\frac{P_o}{hLt}x^2 + \frac{3}{10}\frac{P_o}{hLt}y^2 + \frac{3P_o}{h^3Lt}x^2y^2 - \frac{P_o}{h^3Lt}y^4$$

Discussion

Elementary strength of materials method gives



$$M = \frac{1}{3}x \cdot \frac{P_o x}{L} \cdot \frac{x}{2} = \frac{Px^3}{6L}$$

$$\sigma_x = -\frac{My}{I} = -\frac{Px^3}{6L} \cdot y \cdot \frac{1}{\frac{1}{12}th^3} = -\frac{2P_o}{h^3Lt}x^3y$$

$$R = \frac{\sigma_x\big|_{elasticity}}{\sigma_x\big|_{elementary}} = \frac{\frac{3}{10}\frac{P_o}{t} + \frac{1}{h^2}\frac{L^2P_o}{t} - \frac{1}{2}\frac{P_o}{t}}{\frac{P_o}{t}\left(\frac{L}{h}\right)^2}$$

at $x = L$

$y = -\frac{h}{2}$

if $\frac{L}{h} = 10$, $R = \frac{0.3 + 100 - 0.5}{100} = 99.8\%$

$\frac{L}{h} = 5$ $\quad R = \frac{0.3 + 25 - 0.5}{25} = 99.2\%$

$L/h = 2$ $\quad R = \frac{3.8}{4} = 95\%$