Review for the Mid-Term Exam

Mid-Term Exam

Date: March 13 (Thursday), 2008

Time: 11:00pm-12:20pm

Place: Rm 112, Nedderman Hall

Total Scores: 180 points

Part I: Comprehension: 80 pts

Part 2: Computation: 100 pts

Part 1: 4 problems

- Simple derivation
- Simple computation
- Select stress function

Part 2: 3 problems(2+1HW)

- Take Home (one problem from HW#6)
- In-Class
  - Similar to the homework problems or examples given in the text/lecture notes
  - Each problem may contain several sub-problems.

Exam Contents and Grading

- At least one problem from each chapter
- Equations used for the computation will be provided.
- Generalized Hook’s law will not be given.
- Problems with tedious mathematical manipulation will be avoided.
- Long derivation will not appear in the exam.

Exam Contents and Grading
(continued)

- Partial scores for both derivation and computations will be given.
- Partial scores will be given for providing the procedures of derivation/computation if the time is running out.
- Small bonus points may be given if the engineering judgment is provided for your far-from-make-sense results.
- CHEATING WILL BE REPORTED TO SCHOOL.

Chapter 1

Analysis of Stress

Chapter 1- Review

- Stress Transformation
- Principal Stresses and Maximum Shear stress
- Stress in a given plane
- Stress Invariants
- Octahedral Normal and Shear Stresses

Stress Transformation

$$[\sigma]' = [T]\cdot[\sigma]\cdot[T]^T$$

$$[T] = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$$

3 X 3

$$\begin{bmatrix} \sigma_{x'} \\ \sigma_{y'} \\ \tau_{x'y'} \end{bmatrix} = [T_\sigma(\theta)].\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

$$[T_\sigma(\theta)] = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix}$$

1.8 Principal Stresses in 3-D State of stress

Equation (1-16) gives,

$$\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{pmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} \sigma_p & 0 & 0 \\ 0 & \sigma_p & 0 \\ 0 & 0 & \sigma_p \end{pmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix}$$

$$\therefore \begin{pmatrix} \sigma_p - \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_p - \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_p - \sigma_z \end{pmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} = 0$$

To satisfy the above equation, we have

$$\begin{vmatrix} \sigma_p - \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_p - \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_p - \sigma_z \end{vmatrix} = 0$$

1.8 Prin. Stresses in 3-D State of stress(Cont'd)

The stress invariants $I_1 = I_1'$, $I_2 = I_2'$ and $I_3 = I_3'$

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}{}^2 - \tau_{yz}{}^2 - \tau_{xz}{}^2$$

$$or \qquad = \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{vmatrix} + \begin{vmatrix} \sigma_y & \tau_{yz} \\ \tau_{yz} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_z & \tau_{xz} \\ \tau_{xz} & \sigma_x \end{vmatrix}$$

$$I_3 = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ & \sigma_y & \tau_{yz} \\ sym & & \sigma_z \end{vmatrix}$$

1.10 Octahedral Planes and Stresses(Cont'd)

The normal stresses acting on those planes are identical; so do the shear stresses. There are given as,

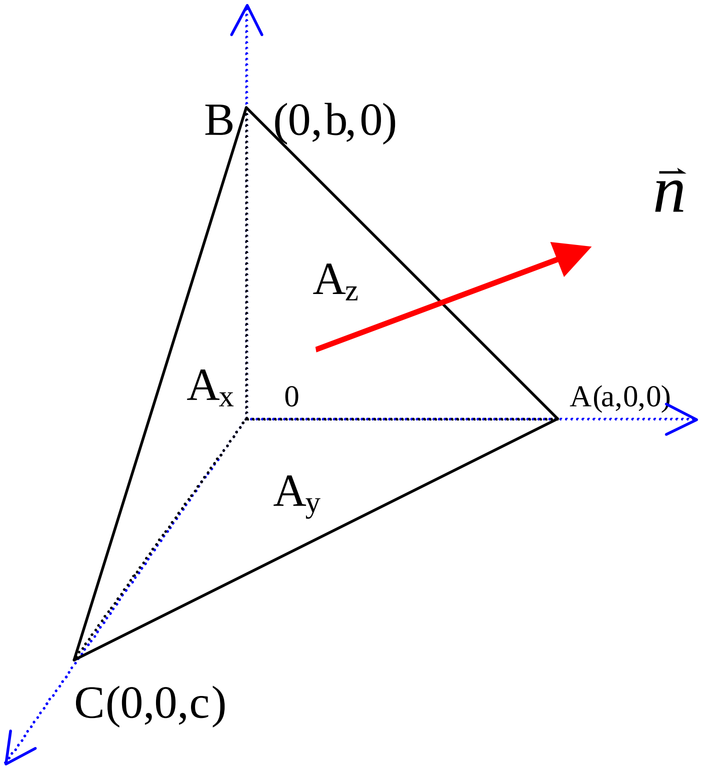
$$\sigma_{oct} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

$$\tau_{oct} = \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{\frac{1}{2}}$$

$$= \frac{2}{3}\left[\tau_{1,2}{}^2 + \tau_{2,3}{}^2 + \tau_{1,3}{}^2\right]^{\frac{1}{2}}$$

$$= \frac{1}{3}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\left(\tau_{xy}{}^2 + \tau_{yz}{}^2 + \tau_{xz}{}^2\right)\right]^{\frac{1}{2}}$$

Directional Cosines of a Given Plane



From vector analysis, we have

$$\vec{n} \qquad \vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k}$$

$$\vec{n}\left\|\vec{A}\right\| = \frac{1}{2}A\vec{B}\times A\vec{C} = Area \quad of \quad \Delta ABC$$

$$A\vec{B} = \pm a\vec{i} \mp b\vec{j}$$

$$A\vec{C} = \pm a\vec{i} \mp c\vec{k}$$

Directional Cosines of a Given Plane(Cont'd)

$$\vec{n}\left\|\vec{A}\right\| = \frac{1}{2}\left(\pm a\vec{i} \mp b\vec{j}\right)\times\left(\pm a\vec{i} \mp c\vec{k}\right) = \frac{1}{2}\begin{vmatrix} i & j & k \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix}$$

$$= \frac{1}{2}\left[a^2\vec{i}\times\vec{i} + (-ab)\vec{j}\times\vec{i} + (-ac)\vec{i}\times\vec{k} + bc\vec{j}\times\vec{k}\right]$$

$$= \frac{1}{2}\left(bc\,\vec{i} + ac\,\vec{j} + ab\,\vec{k}\right)$$

$$\left\|\vec{A}\right\| = \frac{1}{2}\sqrt{(bc)^2 + (ac)^2 + (ab)^2}$$

$$Cos\ \alpha = \frac{A_x}{\|A\|} = \frac{bc}{\left[(bc)^2 + (ac)^2 + (ab)^2\right]^{\frac{1}{2}}} = l$$

$$Cos\ \beta = \frac{A_x}{\|A\|} = \frac{ac}{\left[(bc)^2 + (ac)^2 + (ab)^2\right]^{\frac{1}{2}}} = m$$

$$Cos\ \gamma = \frac{A_x}{\|A\|} = \frac{ab}{\left[(bc)^2 + (ac)^2 + (ab)^2\right]^{\frac{1}{2}}} = n$$

Chapter 2

Strain and Stress-Strain Relationship

Chapter 2- Review

- Strain Transformation
- Principal Strains and Maximum Shear strain
- Rigid Body Motion
- Strain Invariants
- Stress/Strain relationship (including thermal effect)
- Strain Energy
- Stress/Strain with and without contraints

2.2 Strain Transformation

Strain Transformation follows the transformation law (equation 1-9 of stress).

$$[\varepsilon'] = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{pmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{pmatrix} \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \qquad (2\text{-}5)$$

$$[\varepsilon] = [T][\varepsilon][T]^T \qquad (1\text{-}9)$$

2.2.2 2-D Strain Transformation

$$\begin{pmatrix} \varepsilon_x' \\ \varepsilon_y' \\ \gamma_{xy}' \end{pmatrix} = \begin{pmatrix} Cos^2\theta & Sin^2\theta & Sin\theta.Cos\theta \\ Sin^2\theta & Cos^2\theta & -Sin\theta.Cos\theta \\ -2Sin\theta.Cos\theta & 2Sin\theta.Cos\theta & Cos^2\theta - Sin^2\theta \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} \quad (2\text{-}7)$$

$$[\varepsilon_x'] = [T_\varepsilon(\theta)][\varepsilon]$$

Where

$$[T_\varepsilon(\theta)] = \begin{pmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2 - n^2 \end{pmatrix}$$

2.4 Rigid Body Motion

An elastic body undergoes a **motion** without inducing strain is rigid body motion.

Rigid Body Translation

An elastic body undergoes a **displacement** without inducing strain.

$$\left.\begin{aligned} u &= u_0 \\ v &= v_0 \\ w &= w_0 \end{aligned}\right\} \Rightarrow \frac{\partial u}{\partial x} = 0 \qquad \frac{\partial v}{\partial y} = 0 \qquad \frac{\partial w}{\partial z} = 0$$

$$\Rightarrow \varepsilon_x = \varepsilon_y = \varepsilon_z = 0$$

Rigid Body Motion (cont'd)

Rigid Body Rotation

An elastic body undergoes a **rotational displacement** without inducing a shear strain in x-y plane.

$$\gamma_{xy} = 0 \quad \Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \text{Constant.}$$

$$\left.\begin{aligned} u &= ky + C_1(x) \\ v &= -kx + C_2(y) \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \frac{\partial u}{\partial x} &= C_1' = 0 \\ \frac{\partial v}{\partial y} &= C_2' = 0 \end{aligned}\right\}$$

No Translation;

Rotation only

$\Rightarrow C_1' = C_2' = 0$ or C_1 = Constant, C_2 = Constant.

Assuming $C_1 = 0, C_2 = 0$ $\quad u = ky \quad v = -kx$

Linear Elastic Stress-Strain Relation

Total strain is a summation of all strain components in the same direction.

$$\left.\begin{aligned} \varepsilon_x &= \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} \\ \varepsilon_y &= -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} \\ \varepsilon_z &= -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E} \end{aligned}\right\} + \text{No shear strain}$$

or,

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{pmatrix} = \frac{1}{E}\begin{pmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{pmatrix}\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} \qquad (2\text{-}10)$$

2.8.4 Stress-Strain Relationship

Combining equation (2-20) and (2-21), we can write stress-strain relationship for a general solid under mechanical and thermal loads.

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix} + \begin{pmatrix} \alpha \\ \alpha \\ \alpha \\ 0 \\ 0 \\ 0 \end{pmatrix} \Delta T$$

$$[\varepsilon] = \underbrace{[S]\cdot[\sigma]}_{\text{Mechanical Strain}} + \underbrace{[\alpha]\Delta T}_{\text{Thermal Strain}} \qquad [\boldsymbol{\sigma}] = [C]\cdot\{[\boldsymbol{\varepsilon}] - [\boldsymbol{\alpha}]\Delta T\} \qquad \text{(2-22)}$$

Chapter 3

Two Dimensional Problems in Elasticity

Chapter 3- Review

- Plane Stress/Strain
- Airy Stress Function
- Stress/Strain relationship in polar coordinates
- Stress under a concentrated Load

Plane Stress Vs. Plane Strain

Plane stress (3-9)

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\sigma_y\right]$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\sigma_x\right]$$

$$\varepsilon_z = \frac{1}{E}\left[-\nu\left(\sigma_x + \sigma_y\right)\right]$$

$$= -\frac{\nu}{1-\nu}\left(\varepsilon_x + \varepsilon_y\right)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \qquad \gamma_{yz} = \gamma_{xz} = 0$$

Plane Strain (3-3)

$$\varepsilon_x = \frac{1-\nu^2}{E}\left[\sigma_x - \frac{\nu}{1-\nu}\sigma_y\right]$$

$$\varepsilon_y = \frac{1-\nu^2}{E}\left[\sigma_y - \frac{\nu}{1-\nu}\sigma_x\right]$$

$$\varepsilon_z = 0 \Rightarrow \sigma_z = \nu\left(\sigma_x + \sigma_y\right)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \qquad \gamma_{yz} = \gamma_{xz} = 0$$

Plane Stress Vs. Plane Strain

Comparing equation (3-3) and (3-9), we observe:

The equation for a plane stress problem can be converted to a plane strain problem by using E replaced by $\dfrac{E}{1-\nu^2}$ and ν replaced by $\dfrac{\nu}{1-\nu}$.

Plane stress $\longrightarrow$ Plane strain

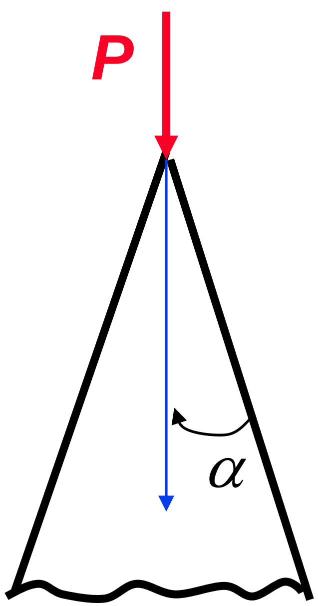
$$E \longrightarrow \frac{E}{1-\nu^2}$$

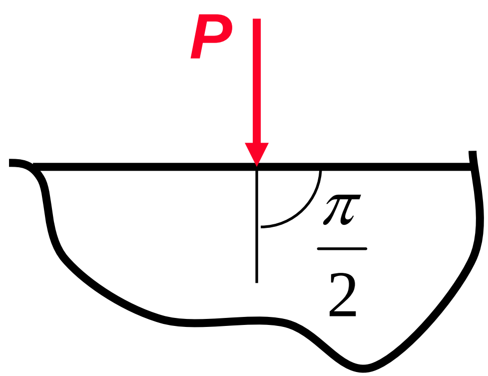
$$\nu \longrightarrow \frac{\nu}{1-\nu} \qquad (3\text{-}11)$$

Stresses Due to Concentrated Load

Hence

$$\left.\begin{aligned}\sigma_r &= -\frac{2\cos\theta}{r}\cdot\frac{P}{2\alpha+\sin 2\alpha}\\ \sigma_\theta &= 0\\ \tau_{r\theta} &= 0\end{aligned}\right\} \qquad (3\text{-}30)$$





If $\alpha = \dfrac{\pi}{2}$

$$\sigma_r = -\frac{2P\cos\theta}{\pi . r} \qquad (3\text{-}31)$$

Chapter 4

Criteria of Material Failure

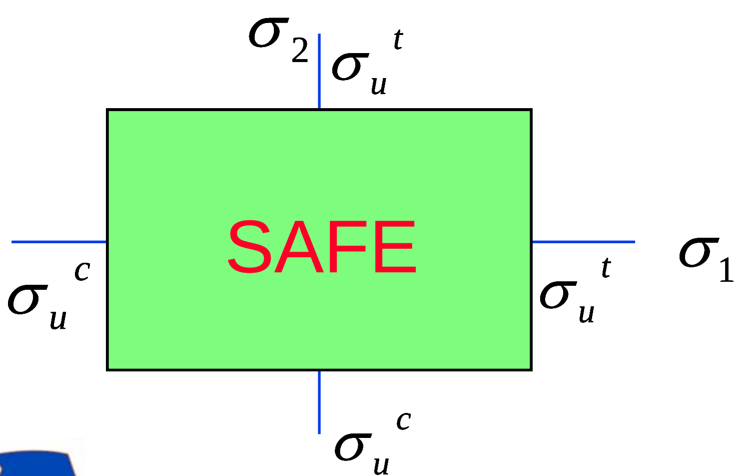
Chapter 4- Review

- Failure Criteria and their Failure Envelop
- Max. Principal Stress
- Max. Shear Stress
- Coulomb –Mohr
- Max. Distortion Energy (Von-Mises)
- Max. Allowable Stress and Safety Factor

Maximum Principal Stress Criterion

* No effect of σ_1 and σ_2 interaction

Failure envelop



$$\sigma_3 = 0$$

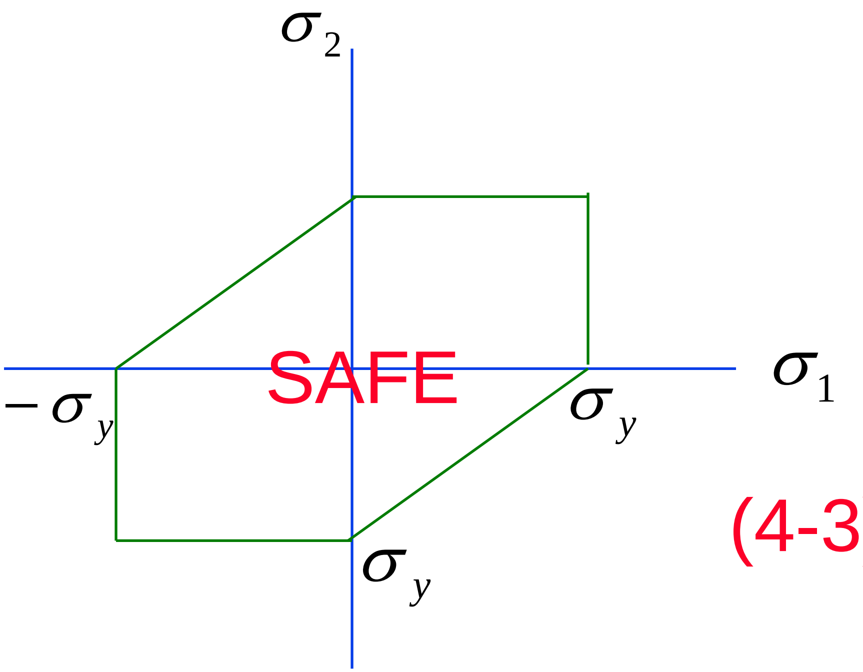
Maximum Shear Stress Criterion

Failure envelop

$$|\sigma_1 - \sigma_2| \geq |\sigma_y|$$

$$|\sigma_1| \geq |\sigma_y|$$

$$|\sigma_2| \geq |\sigma_y|$$

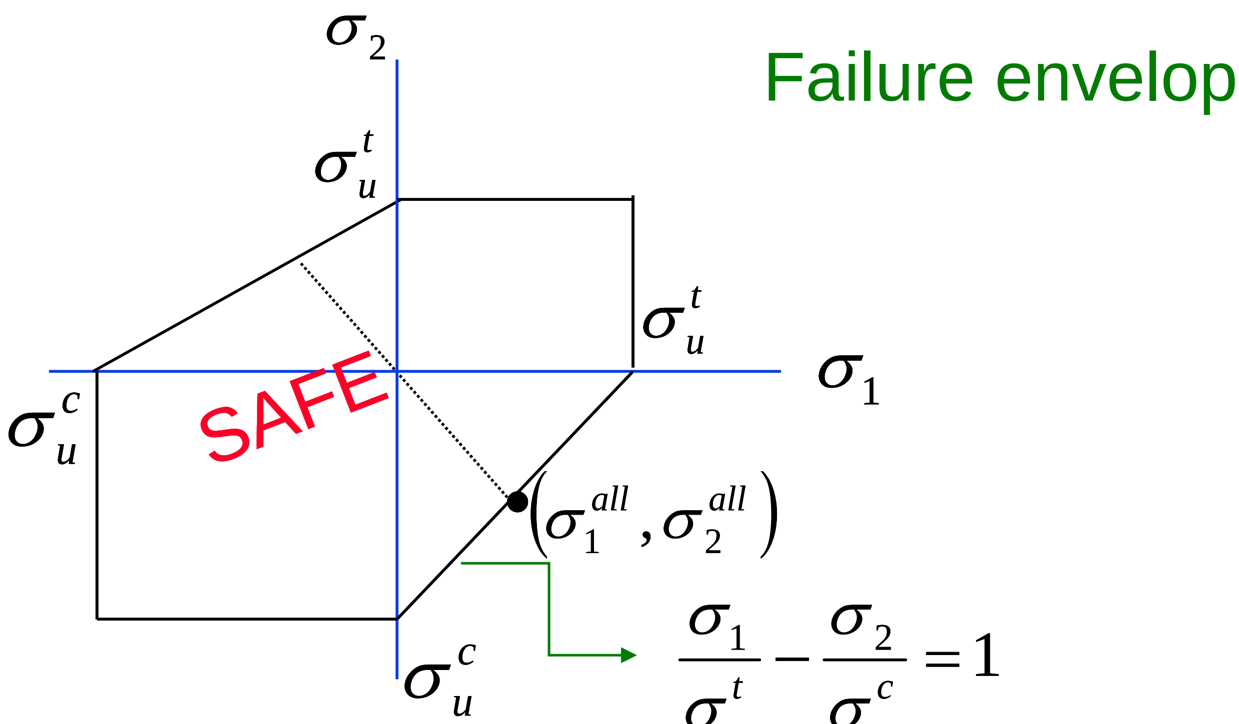


(4-3)

Coulomb – Mohr Failure Criterion

In general, $\sigma_u^t \neq \sigma_u^c$

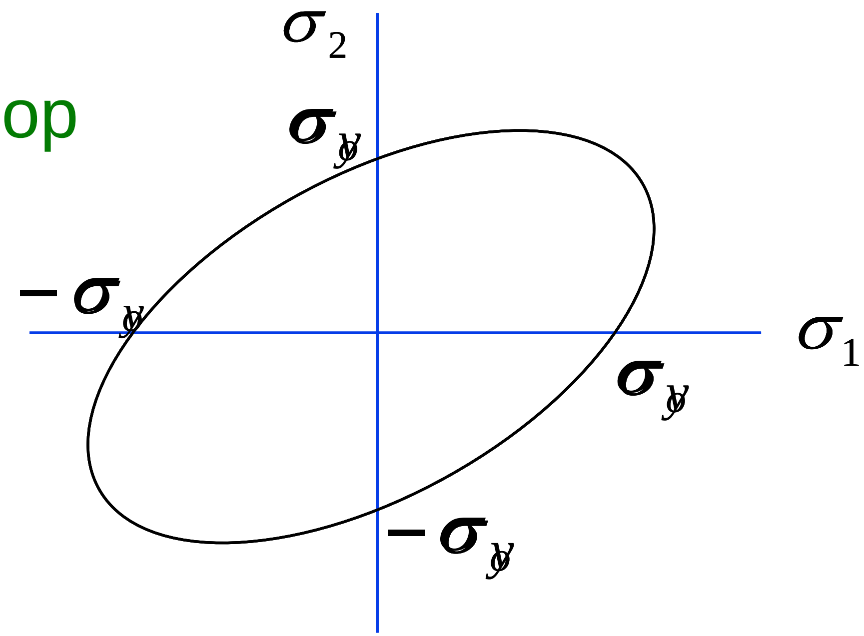
The Coulomb – Mohr failure criterion is given as



$$\frac{\sigma_1}{\sigma_u^t} - \frac{\sigma_2}{\sigma_u^c} = 1$$

Maximum Distortion Energy Criterion

Failure envelop



$$\sigma_1^{\ 2} + \sigma_2^{\ 2} - \sigma_1\sigma_2 \geq \sigma_y^{\ 2}$$

Chapter 5

Beam Bending

Chapter 5- Review

- Pure Bending of Beams with sym./asym. Cross-section (no shear)
- Bending of cantilever and simply supported Beams with narrow cross-section
- Elementary Beam Theory
- Transverse Shear and shear flow
- Composite Beam (Beam made of multiple isotropic materials bonded together)

$$\sigma_x = \frac{M_y I_{yz} + M_z I_y}{I_{yz}^2 - I_y I_z}.y - \frac{M_z I_{yz} + M_y I_z}{I_{yz}^2 - I_y I_z}.z \qquad (5\text{-}10)$$

The equation of neutral axis is located at $\sigma_x = 0$, i.e.

$$\frac{y}{z} = \frac{M_y I_z + M_z I_{yz}}{M_y I_{yz} + M_z I_y} = \tan\Phi \qquad (5\text{-}11)$$

Equation (5-11) can be simplified by rotating a coordinate system such that

$$\frac{y'}{z'} = \frac{M_y'}{M_z'}\frac{I_z'}{I_y'} \quad \text{and} \quad \sigma_x = -\frac{M_z'}{I_z'}.y' + \frac{M_y'}{I_y'}.z' \qquad (5\text{-}12)$$

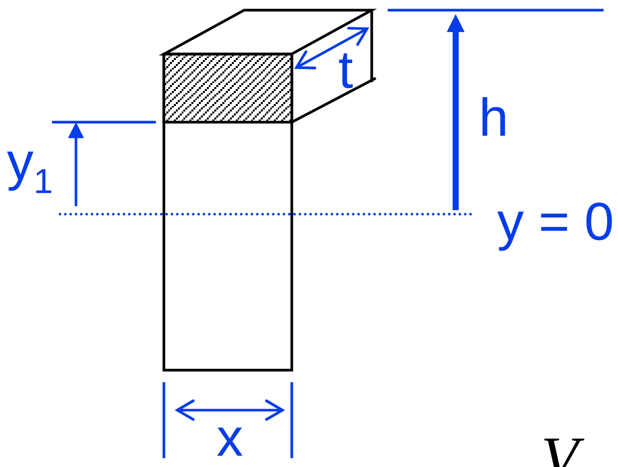
Elementary Theory of Beam

$$\left.\begin{array}{ll} \dfrac{dV}{dx} = -P & \dfrac{dM_z}{dx} = -V \\[2ex] \dfrac{d^2 M_z}{dx^2} = P & \dfrac{d^2}{dx^2}\left(EI_z \dfrac{d^2 v}{dx^2} \right) = P \end{array}\right\} \quad (5\text{-}23)$$

The displacement function, v can be obtained from the direct integration of the above function.

General Symmetrical Cross Section

For a rectangular cross – section, $t = h$



$$\bar{y} = y_1 + (h - y_1)/2 = \frac{h + y_1}{2}$$

$$A^* = (h - y_1).t$$

$$Q = A^* \bar{y} = \frac{h + y_1}{2}(h - y_1).t = \frac{h^2 - y_1^2}{2}t$$

$$\tau_{xy} = \frac{V}{I_z t}.\frac{h^2 - y_1^2}{2}t = \frac{V}{2I_z}\left(h^2 - y_1^2\right) \qquad \text{(5-25)}$$

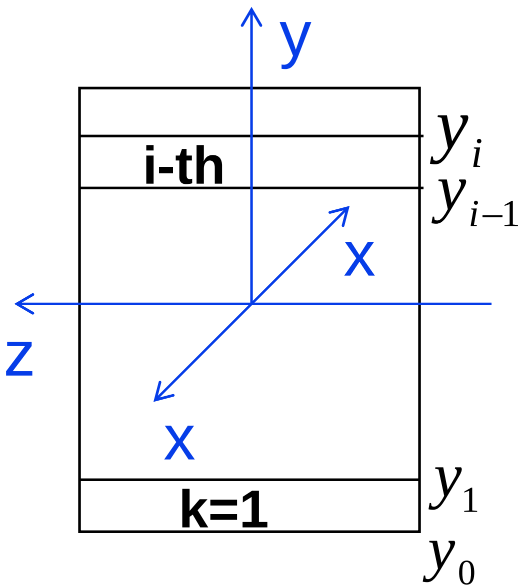
$$\tau_{xy}\Big|_{Max \quad at \quad y_1=0} = \frac{V}{2I_z}.h^2$$

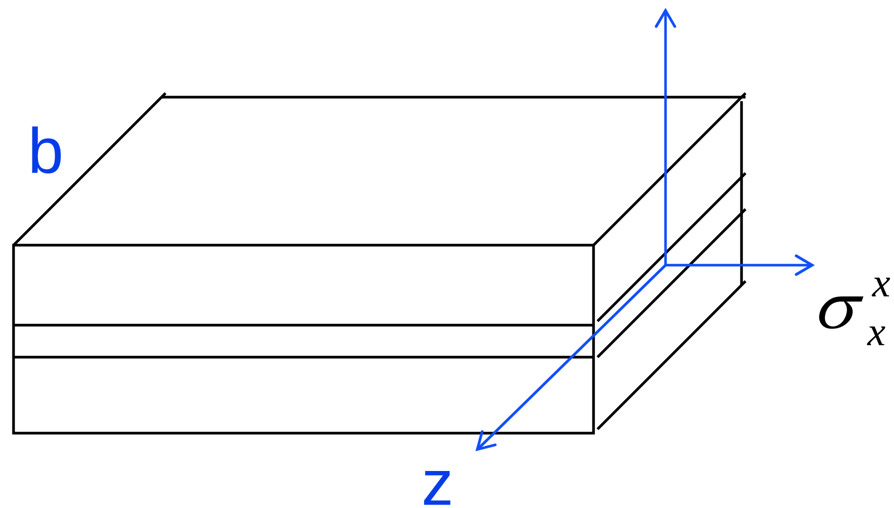
$$= \frac{Vh^2}{2\frac{2}{3}h^3 t} = \frac{3V}{4ht} = \frac{3}{2}\frac{V}{A}$$

$$I_z = \frac{1}{12}(t).(2h)^3 = \frac{2}{3}h^3 t$$

$$2h.t = A$$

Composite Beams





$$M_z = -\int_{-h}^{h} \sigma_x y dA = -\sum_{i=1}^{h} \int_{y_{i-1}}^{y_i} \sigma_x^i y b dA$$

$$\sigma_x^i = E_i \varepsilon_x^i \qquad \varepsilon_x^i = \underbrace{\varepsilon_x^0}_{mid-plane\ \ strain} + y\overbrace{k}^{curvature}$$

Composite Beams (Cont'd)

$$M_z = -\sum_{i=1}^{n} \int_{y_{i-1}}^{y_i} E_i \left(\varepsilon_x^0 + yk \right) y dy$$

$$= \left[-\frac{1}{2} \varepsilon_x^0 \sum_{i=1}^{n} E_i \left(y_1^2 - y_{i-1}^2 \right) - \frac{k}{3} \sum_{i=1}^{n} E_i \left(y_1^3 - y_{i-1}^3 \right) \right] . b$$

$$M = -B\varepsilon_x^0 - Dk$$

Where,

$$B = \frac{b}{2} \sum_{i=1}^{n} E_i \left(y_1^2 - y_{i-1}^2 \right) \qquad \text{(5-28)}$$

$$D = \frac{b}{3} \sum_{i=1}^{n} E_i \left(y_1^3 - y_{i-1}^3 \right) \qquad \text{(5-28a)}$$

Composite Beams (Cont'd)

$$P = \int_{-h}^{h} \sigma_x dA = \sum_{i=1}^{n} \int_{y_{i-1}}^{y_i} E_i(\varepsilon_0 + yk) b dy$$

$$P = \varepsilon_0 \sum_{i=1}^{n} \int_{y_{i-1}}^{y_i} E_i b dy + k \sum_{i=1}^{n} \int_{y_{i-1}}^{y_i} E_i b y dy = \varepsilon_0 A + kB \quad \text{(5-29)}$$

Where,

$$A = \sum_{i=1}^{n} \int_{y_{i-1}}^{y_i} E_i b dy = \sum_{i=1}^{n} E_i b(y_i - y_{i-1}) = \sum_{i=1}^{n} E_i A_i$$

$$B = b \sum_{i=1}^{n} \int_{y_{i-1}}^{y_i} E_i y dy = \frac{b}{2} \sum_{i=1}^{n} E_i (y_i^2 - y_{i-1}^2)$$

(5-29a)

$$\begin{bmatrix} P \\ -M \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} \varepsilon_0 \\ k \end{bmatrix} \quad \text{(5-30)}$$

Chapter 6

Axi symmetrically Loaded Members

Definition

A structure under load exhibiting a symmetrical stress distribution with respect to an axis is called as " axisymmetrically loaded member". Assuming z-axis to be the symmetrical axis, then we can conclude the stresses to be independent of θ. This implies no displacement in the θ direction, no shear stress in the rθ-plane. That is,

$$\begin{cases} V_\theta = 0 \\ \tau_{r\theta} = 0 \end{cases}$$

In this chapter, we assume $\sigma_z = 0$ if there is no axial load.

6.5 Stress Components of Loaded Members

$$\varepsilon_r = \frac{du}{dr} = C_1 - \frac{C_2}{r^2} \qquad \varepsilon_\theta = \frac{u}{r} = C_1 + \frac{C_2}{r^2} \qquad (6\text{-}6)$$

Substituting (6-6) into (6-2),

$$\sigma_r = \frac{E}{1-\nu^2}\left[\left(C_1 - \frac{C_2}{r^2}\right) + \nu\left(C_1 + \frac{C_2}{r^2}\right)\right] = C_1\frac{E}{1-\nu} - C_2\frac{E}{1+\nu}.\frac{1}{r^2}$$

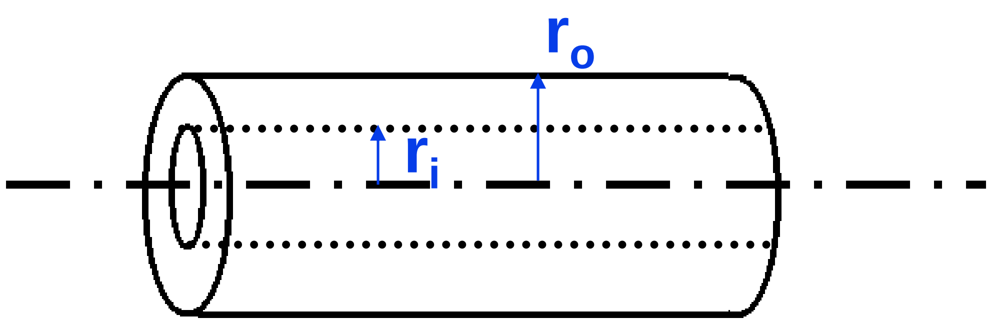
$$= K_1 - \frac{K_2}{r^2}$$

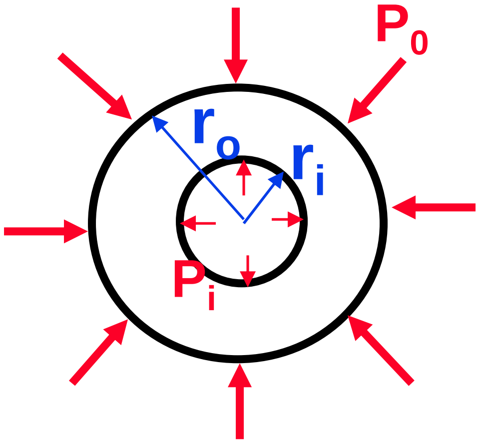
$$\sigma_\theta = \frac{E}{1-\nu^2}\left[\left(C_1 + \frac{C_2}{r^2}\right) + \nu\left(C_1 - \frac{C_2}{r^2}\right)\right] = C_1\frac{E}{1-\nu} + C_2\frac{E}{1+\nu}.\frac{1}{r^2}$$

$$= K_1 + \frac{K_2}{r^2}$$

$$\boxed{\sigma_r = K_1 - \frac{K_2}{r^2}}, \quad \boxed{\sigma_\theta = K_1 + \frac{K_2}{r^2}} \qquad (6\text{-}7)$$

6-6 Stresses in Pressurized Cylinder





$$o \rightarrow outer$$
$$i \rightarrow inner$$

B.C.

$$\sigma_r = -P_0 \qquad at \quad r = r_0$$
$$\sigma_r = -P_i \qquad at \quad r = r_i$$

Stresses in Pressurized Cylinder (Cont'd)

$$\left.\begin{aligned}\sigma_r &= K_1 - \frac{K_2}{r_o^2} = -P_o \\ \sigma_r &= K_1 - \frac{K_2}{r_i^2} = -P_i\end{aligned}\right\} \Rightarrow \begin{aligned}K_1 &= \frac{P_i r_i^2 - P_o r_o^2}{r_o^2 - r_i^2} \\ K_2 &= \frac{(P_i - P_o) r_i^2 r_o^2}{r_o^2 - r_i^2}\end{aligned} \qquad (6\text{-}8)$$

For rotating cylinder with angular velocity, ω

$$F_r = \rho . r \omega^2$$

$$\rho \rightarrow mass \quad density$$

Case 1: No Rotation and No External Pressure

$$K_1 = \frac{P_i r_i^2 - P_o r_o^2}{r_o^2 - r_i^2} \qquad K_2 = \frac{(P_i - P_o) r_i^2 r_o^2}{r_o^2 - r_i^2} \qquad \text{(6-8)}$$

$$F_r = 0 \quad \& \quad P_0 = 0 \qquad K_1 = \frac{P_i r_i^2}{r_o^2 - r_i^2} \qquad K_2 = \frac{P_i r_i^2 r_o^2}{r_o^2 - r_i^2} = r_o^2 K_1$$

$$\sigma_r = K_1 - \frac{K_2}{r^2} \qquad \sigma_\theta = K_1 + \frac{K_2}{r^2} \qquad \text{(6-7)}$$

$$\sigma_r = K_1 - \frac{K_2}{r^2} = K_1\left[1 - \frac{r_o^2}{r^2}\right]$$

$$r_i \le r \le r_o \qquad \text{(6-9)}$$

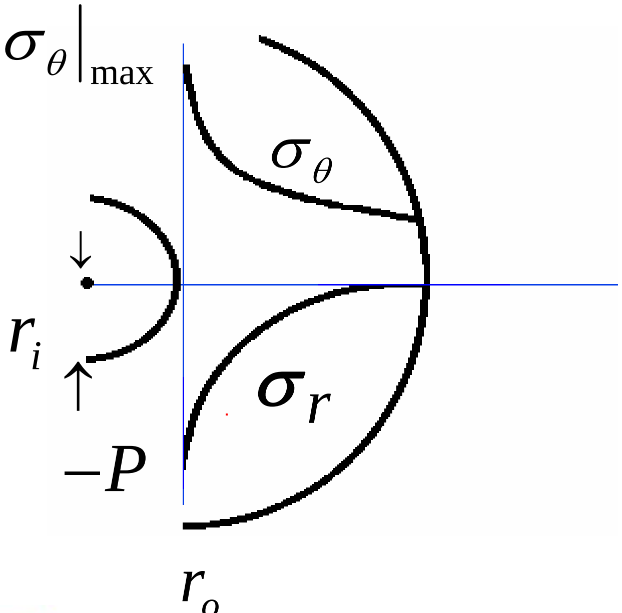
$$\sigma_\theta = K_1 + \frac{K_2}{r^2} = K_1\left[1 + \frac{r_o^2}{r^2}\right]$$

Case 1: (Cont'd)

Maximum stress occurs at $r = r_o$

$$\sigma_r = K_1 - \frac{K_2}{r^2} = K_1\left[1 - \frac{r_o^2}{r^2}\right] \qquad \sigma_\theta = K_1 + \frac{K_2}{r^2} = K_1\left[1 + \frac{r_o^2}{r^2}\right]$$

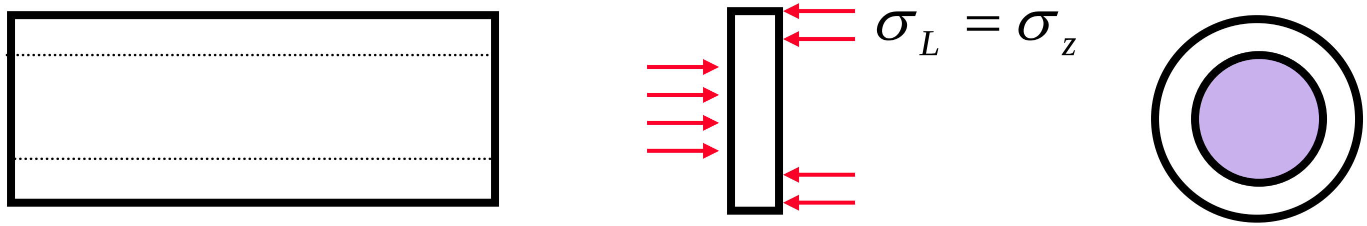
$$\sigma_\theta\big|_{\max} = K_1\left[1 + \left(\frac{r_o}{r_i}\right)^2\right] = P_i \cdot \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2}$$



$$K_1 = \frac{P_i r_i^2 - P_o r_o^2}{r_o^2 - r_i^2}$$

$$K_2 = \frac{(P_i - P_o) r_i^2 r_o^2}{r_o^2 - r_i^2} \qquad (6\text{-}8)$$

$$\sigma_r\big|_{\min} = -P \quad at \quad r = r_i$$



Force acting on the end cap

$$F = \sigma . A = \sigma_z \pi \left(r_o^2 - r_i^2 \right) = P_i \pi r_i^2 \qquad (6\text{-}10)$$

$$\sigma_z = P_i \frac{r_i^2}{r_o^2 - r_i^2}$$

Case 3: Rotating Cylinder (Cont'd)

The stresses have the form

$$\sigma_r = K_1 - \frac{K_2}{r^2} - K_3 r^2$$

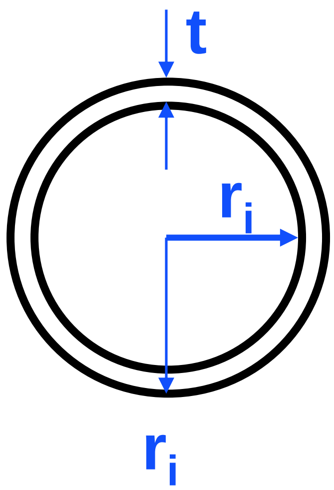
$$\sigma_\theta = K_1 + \frac{K_2}{r^2} - K_4 r^2$$

(6-11)

$$K_1 = \frac{P_i r_i^2 - P_o r_o^2}{r_o^2 - r_i^2} \qquad K_2 = \frac{(P_i - P_o) r_i^2 r_o^2}{r_o^2 - r_i^2}$$

$$K_3 = \frac{3+\nu}{8}\rho\omega^2 \qquad K_4 = \frac{1+3\nu}{8}\rho\omega^2$$

6.7 Thin-Walled Vessels (No External Pressure)



$$t = r_o - r_i \quad \text{(wall thickness)}$$

$$\frac{t}{r_i} << \frac{1}{20} \qquad \frac{r_i}{r_o} \approx 1$$

$$K_1 = \frac{P_i r_i^2}{r_o^2 - r_i^2} \quad (P_o = 0)$$

$$= \frac{P_i r_i^2}{(r_o - r_i)(r_o + r_i)} = \frac{P_i r_i^2}{2.r_i.t}$$

$$= \frac{P_i r_i}{2t}$$

$$\sigma_r = K_1 - \frac{K_2}{r^2}$$

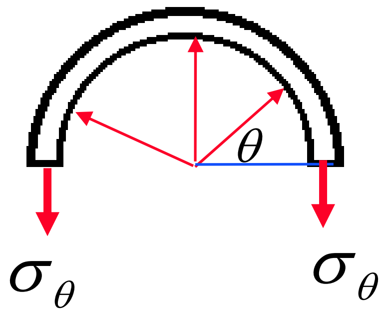
$$K_2 = \frac{P_i r_i^2 r_o^2}{r_o^2 - r_i^2}$$

$$= r_o^2 K_1$$

Thin-Walled Vessels (Cont'd)

$$\sigma_r = K_1\left(1-\frac{r_o^2}{r^2}\right) \approx 0$$

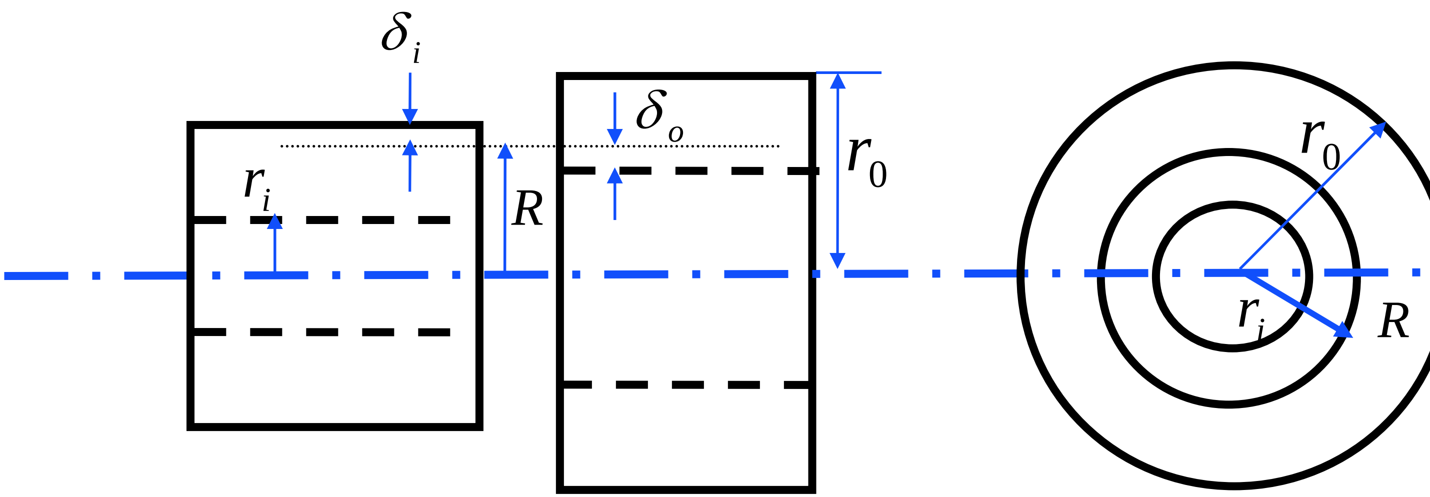
$$\sigma_\theta = K_1 + \frac{K_2}{r^2} = K_1\left(1+\frac{r_o^2}{r^2}\right) = 2K_1 = \frac{P_i r_i}{t} \qquad \text{(6-12)}$$



$$\int_0^\pi P_i \sin\theta . r_i d\theta = \sigma_\theta . t . 2$$

$$\sigma_\theta = \frac{P_i r_i}{t}$$

Press and Shrink Fits (Cont'd)



$$\varepsilon_\theta^i = \frac{2\pi R - 2\pi\left(R+\delta_i\right)}{2\pi\left(R+\delta_i\right)} = \frac{-\delta_i}{R+\delta_i} = -\frac{\delta_i}{R} \qquad \text{(6-16)}$$

Total mismatch $|\delta|$

$$|\delta| = |\delta_o| + |\delta_i| = |\varepsilon_\theta^o R| + |\varepsilon_\theta^i R|$$

$$= \frac{PR}{E_o}\left[\frac{R^2 + r_o^2}{r_o^2 - R^2} + \nu_o\right] + \frac{PR}{E_i}\left[\frac{R^2 + r_i^2}{R^2 - r_i^2} - \nu_i\right] \qquad \text{(6-17)}$$

Press and Shrink Fits (Cont'd)

If $E_o = E_i = E$ and $\nu_o = \nu_i = \nu$

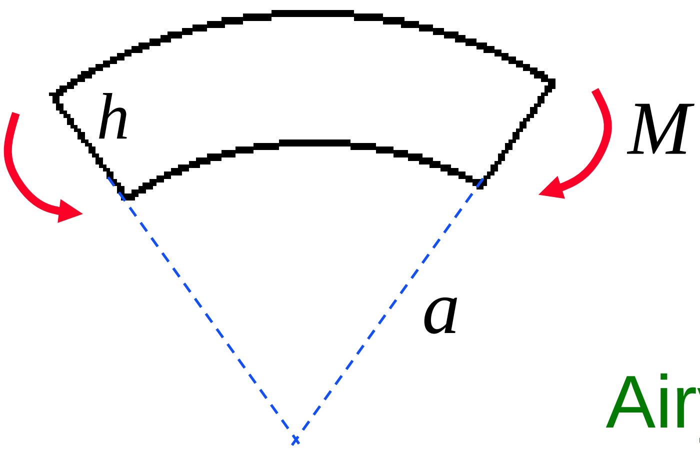
(Two cylinders having the same material)

$$\delta = \frac{PR}{E}\left[\frac{R^2 + r_o^2}{r_o^2 - R^2} + \frac{R^2 + r_i^2}{R^2 - r_i^2}\right] = \frac{PR}{E}\frac{2R\left(r_o^2 - r_i^2\right)}{\left(r_o^2 - R^2\right)\left(R^2 - r_i^2\right)} \qquad (6\text{-}18)$$

If δ is given, then the contact pressure, P can be given as

$$P = \frac{E\delta}{R} \cdot \frac{\left(r_o^2 - R^2\right)\left(R^2 - r_i^2\right)}{2R\left(r_o^2 - r_i^2\right)} \qquad (6\text{-}19)$$

6.9 Curved Beam



Stresses are independent of θ

$$\Rightarrow \tau_{r\theta} = -\frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} = 0$$

Airy Stress Function

$$\sigma_r = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}$$

$$\sigma_\theta = \frac{\partial^2 \Phi}{\partial r^2}$$

$$\sigma_r + \sigma_\theta = \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = \nabla^2 \Phi$$

Curved Beam (Cont'd)

$$r\frac{d\sigma_r^P}{dr}+2\sigma_r^P=r\left(C_2.\frac{1}{r}\right)+2C_1+2C_2\ln r$$

$$=C'\ln r+C'' \qquad \Rightarrow C'=2C_2$$

$$C''=C_2+2C_1$$

$$\sigma_r=\sigma_r^P+\sigma_r^H=C_1+C_2\ln r+\frac{C_3}{r^2} \qquad \text{(6-25)}$$

Form equation (6-22)

$$\sigma_\theta=C'\ln r+C''-\sigma_r$$

$$=2C_2\ln r+C_2+2C_1-C_1-C_2\ln r-\frac{C_3}{r^2}$$

$$=C_1+C_2(1+\ln r)-\frac{C_3}{r^2}$$

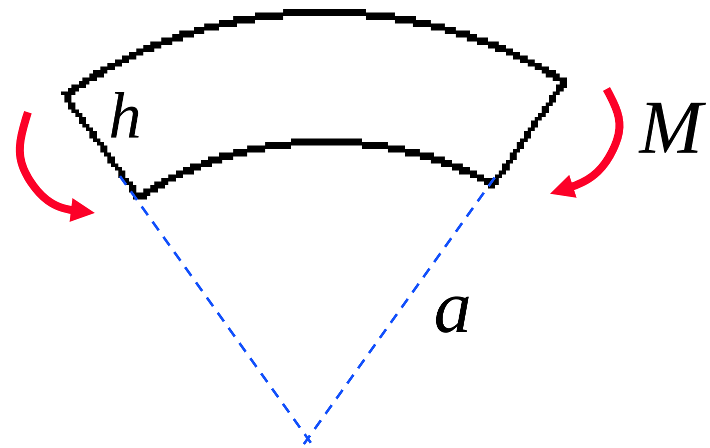
Rewriting,

$$\sigma_r=C_1+C_2\ln\frac{r}{a}+\frac{C_3}{r^2} \qquad \sigma_\theta=C_1+C_2\left(\ln\frac{r}{a}+1\right)-\frac{C_3}{r^2} \qquad \text{(6-26)}$$

Curved Beam (Cont'd)

B.C.

i) $\sigma_r\big|_{r=a} = \sigma_r\big|_{r=b} = 0$



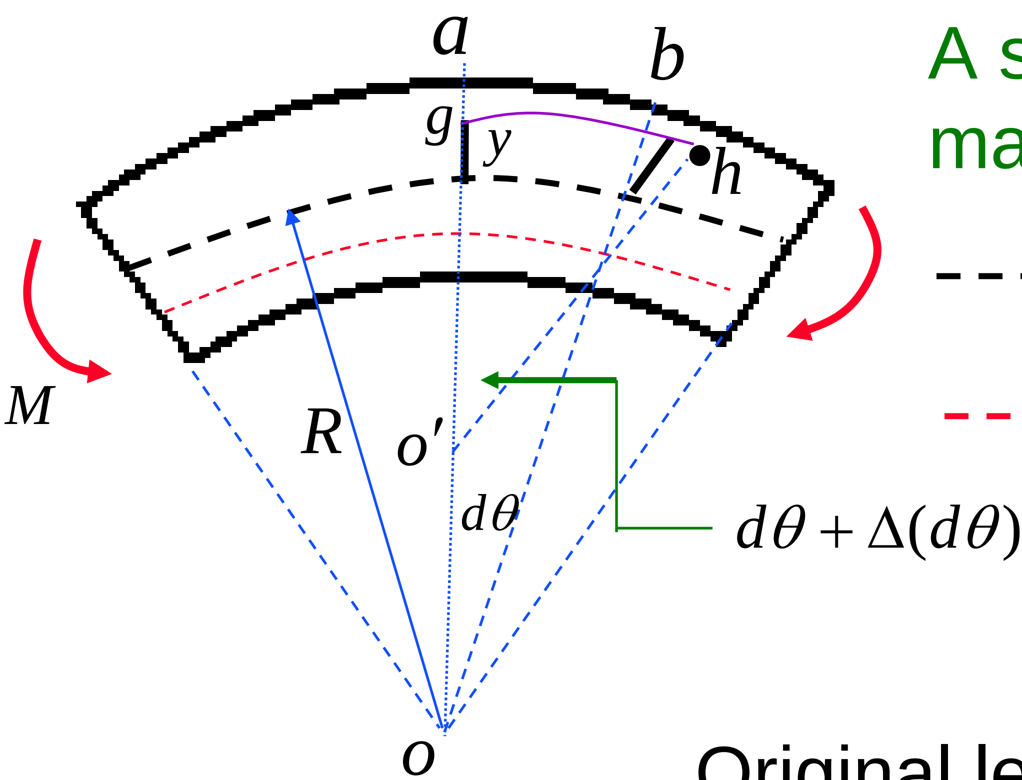
$$\sigma_r\big|_{r=a} = C_1 + \frac{C_3}{a^2} = 0$$

$$C_3 = -C_1 a^2$$

$$\sigma_r\big|_{r=b} = C_1 + C_2 \ln\frac{b}{a} + \frac{C_3}{b^2} = C_1\left(1 - \frac{a^2}{b^2}\right) + C_2 \ln\frac{b}{a} = 0$$

$$C_1 = \frac{b^2}{a^2 - b^2}\ln\frac{b}{a}.C_2 \qquad C_3 = -\frac{a^2 b^2}{a^2 - b^2}\ln\frac{b}{a}.C_2$$

6.10 Winkler's Theory



A similar to strength of material approach

- - - - Centroidal Line

- - - - Neutral Axis

Original length $= (R + y)d\theta$

Deformed length$= \varepsilon_c .Rd\theta + y\Delta(d\theta)$

Winkler's Theory (Cont'd)

$$\lambda = \frac{1}{EA}\left[\frac{M}{R} + \frac{M}{ZR}\right]$$

$$\sigma_{\theta} = \frac{M}{AR}\left[1 + \frac{y}{Z(R+y)}\right]$$

Winkler's Formula

At Neutral Axis, $\Rightarrow \sigma_c = 0$

$$1 + \frac{y_n}{Z(R+y_n)} = 0 \quad \Rightarrow y_n = -\frac{ZR}{Z+1}$$