MAE3315
Homework No.1 (SOLUTION)

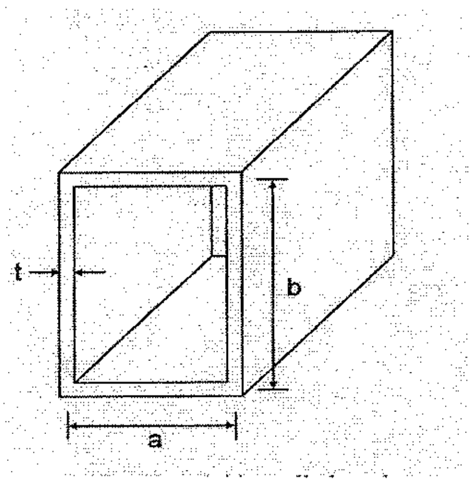
Problem 1.1 (Prob.1.1 in the textbook, P 17)

The beam of a rectangular thin-walled section (i.e., t is very small) is designed to carry both bending moment, M and torque, T. If the total wall contour length L=2(a+b) is fixed, find the optimum b/a ratio to achieve the most efficient section if

$\sigma_{allowable} = 2\ \tau_{allowable}$ $\qquad\qquad$ $M=T$

Note that for closed thin-walled sectioned such as the one shown in the figure, the shear stress due to torsion

$$\tau = \frac{T}{2abt}$$



Solution:

The inertia of the cross-section is,

$$I = \left[\frac{1}{12}(a+t)(b+t)^3\right] - \left[\frac{1}{12}(a-t)(b-t)^3\right]$$

Expanding and rearranging it,

$$I = \frac{t}{6}\left[3ab^2 + at^2 + b^3 + 3bt^2\right]$$

Remembering that t is very small, so t^2 can be neglected.

$$I = \frac{t}{6}\left[3ab^2 + b^3\right]$$

Substituting, $b = \frac{L}{2} - a$,

$$I = \frac{t}{6}\left[3a\left(\frac{L}{2} - a\right)^2 + \left(\frac{L}{2} - a\right)^3\right]$$

Now, $$\sigma = \frac{M \cdot c}{I} = \frac{M \cdot \left(\frac{b+t}{2}\right)}{\frac{t}{6}\left[3a\left(\frac{L}{2} - a\right)^2 + \left(\frac{L}{2} - a\right)^3\right]} = \frac{\frac{M}{2} \cdot \left[\frac{L}{2} - a + t\right]}{\frac{t}{6}\left[3a\left(\frac{L}{2} - a\right)^2 + \left(\frac{L}{2} - a\right)^3\right]}$$

And, taking derivative of the normal stress with respect to "a"; and equating it to zero to find out the maximum, $\frac{\delta\sigma}{\delta a} = 0$

$$\frac{\delta\sigma}{\delta a} = \frac{24\left(-10La + L^2 + 16a^2 + 24at\right)M}{(4a + L)^2(-L + 2a)^3 t} = 0$$

Again, t is very small compared to a and L. Then,

$$16a^2 - 10La + L^2 = 0$$

Solving it,

$$a = \frac{1}{8}L \qquad \text{and} \qquad a = \frac{1}{2}L$$

$a = \frac{1}{2}L$ is discarded; otherwise, b would be 0. So the only solution is,

$$a = \frac{1}{8}L \qquad \text{and} \qquad b = \frac{3}{8}L$$

On the other hand,

$$\tau = \frac{T}{2abt} = \frac{T}{2a\left(\frac{L}{2} - a\right)t}$$

Once again, to get the maximum, $\frac{\delta\tau}{\delta a} = 0$,

$$\frac{\delta\tau}{\delta a} = \frac{T(-L+4a)}{a^2(-L+2a)^2 t} = 0$$

$$4a - L = 0$$

$$a = \frac{L}{4} \qquad \text{and} \qquad b = \frac{L}{4}$$

Now let's consider the 2 cases:

<u>Case I:</u> $a = \frac{1}{8}L$ and $b = \frac{3}{8}L$:

$$\sigma_{MAX} = \frac{M \cdot c}{I} = \frac{M\left(\frac{b+t}{2}\right)}{\frac{t}{6}\left[3ab^2 + b^3\right]} = \frac{3Mb}{t\left[3ab^2 + b^3\right]} = \frac{32}{3}\frac{M}{tL^2}$$

and

$$\tau_{MAX} = \frac{T}{2abt} = \frac{32}{3}\frac{T}{tL^2}$$

Therefore, for this case,

$$\sigma_{MAX} = \tau_{MAX}$$

On the other hand,

$$\tau_{allowable} = \frac{1}{2}\sigma_{allowable}$$

$$\tau_{allowable} = \frac{1}{2}\sigma_{MAX} < \sigma_{MAX}$$

Therefore, when σ_{MAX} is reached, $\tau_{allowable}$ is only half of its potential (50%). That is, it needs ½ of its value to reach its maximum.

<u>*Case II:*</u> $a = \frac{L}{4}$ and $b = \frac{L}{4}$:

$$\sigma_{MAX} = \frac{M \cdot c}{I} = \frac{M\left(\frac{b+t}{2}\right)}{\frac{t}{6}\left[3ab^2 + b^3\right]} = \frac{3Mb}{t\left[3ab^2 + b^3\right]} = 12\frac{M}{tL^2}$$

and

$$\tau_{MAX} = \frac{T}{2abt} = 8\frac{T}{tL^2}$$

Therefore, for this case,

$$\sigma_{MAX} = \frac{3}{2}\tau_{MAX} \qquad \text{or} \qquad \frac{2}{3}\sigma_{MAX} = \tau_{MAX}$$

On the other hand,

$$\tau_{allowable} = \frac{1}{2}\sigma_{allowable}$$

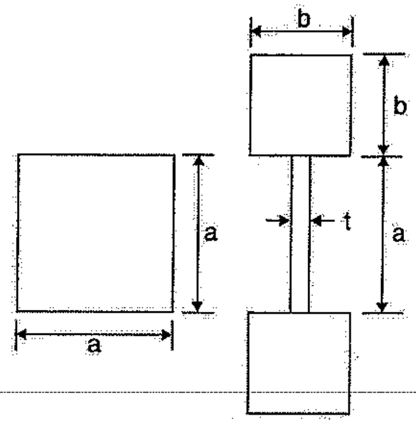
$$\tau_{allowable} = \frac{1}{2}\sigma_{MAX} < \frac{2}{3}\sigma_{MAX}$$

Therefore, when σ_{MAX} is reached, $\tau_{allowable}$ is 5/6 of its potential. That is, it needs only 1/6 of its value to reach its maximum. Therefore, this is the most efficient configuration. So,

$$\left(\frac{b}{a}\right)_{MOST\_EFFICIENT\_RATE} = 1$$

Problem 1.2 (Prob.1.7 in the textbook, P 18)

Compare the load-carrying capabilities of two beams having the respective cross-section shown below. Use bending rigidity as the criterion for comparison. It is given that a=4cm, t=0.2 cm, and the cross-section have the same area.



Solution:

Assuming both beams are made out of the same material.

$$Area_1 = Area_2$$

$$a^2 = 2(b^2) + at$$

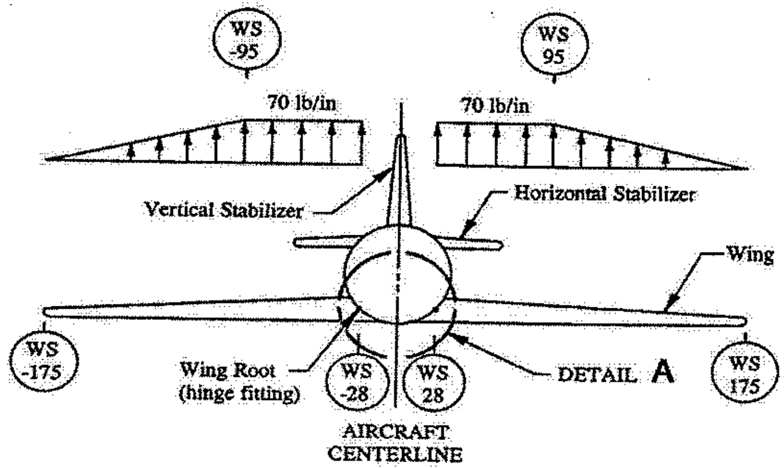
$$b = \sqrt{\frac{a^2 - at}{2}} = 2.757cm$$

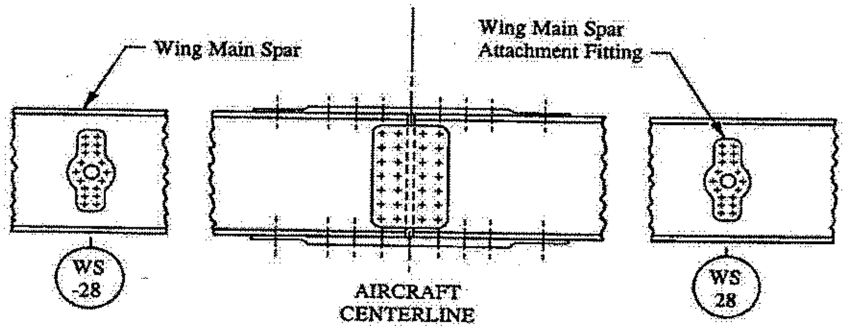
Having in mind that higher bending rigidity means less bending stress, let's compare their bending rigidities,

$(EI)_1$	$(EI)_2$
$E\frac{1}{12}aa^3$	$2\left[\frac{1}{12}bb^3 + b^2\left(\frac{b+a}{2}\right)^2\right] + \frac{1}{12}ta^3$
$(21.33cm^4)E$	$(184.18cm^4)E$

Problem 1.3

An aircraft as shown is designed to carry all of the wind lift forces. For the simplified wing up-bending loads as shown in the figure, draw the shear and moment diagrams of the wing spars.

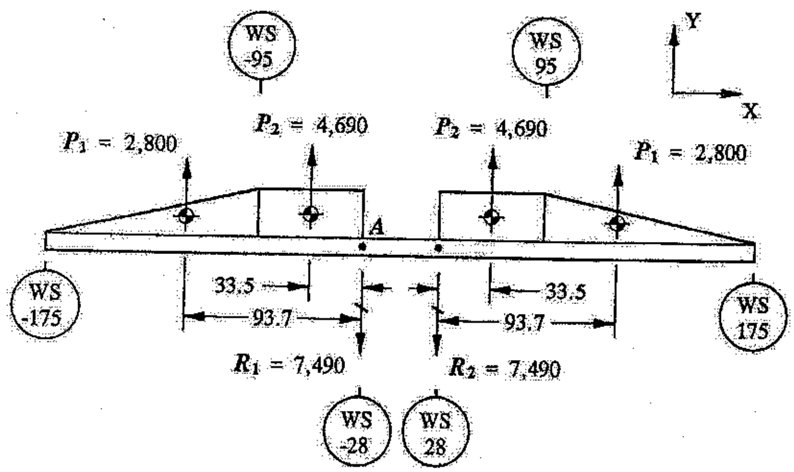




DETAIL A

Solution:

The free-body diagram,



Equivalent concentrated forces for the aerodynamic loading are computed as follows:

$$P_1 = \frac{1}{2} wL = \frac{1}{2}(70)(175-95) = 2{,}800 lb$$

$$P_2 = wL = (70)(95-28) = 4{,}690 lb$$

By symmetry of loading, the wing root reactions are determined by inspection. Essentially, for the wing structure to remain in a state of equilibrium, the following equilibrium equations must be satisfied:

$$\sum F_y = 0 \quad \text{and} \quad \sum M_z = 0.$$

In this example,

$$\sum F_y = 0, \qquad P_1 + P_2 - R_1 - R_2 + P_2 + P_1 = 0.$$

Substituting for P_1 and P_2,

$$2{,}800 + 4{,}690 - R_1 - R_2 + 4{,}690 + 2{,}800 = 0$$

$$-R_1 - R_2 = -14{,}980$$

From $\sum M_z = 0$ at point A, R_2 is obtained directly:

$$\sum M_A = 0 \quad 4{,}690(33.5) + 2{,}800(93.7) + R_2(56.0) - 4{,}690(89.5) - 2{,}800(149.7) = 0$$

$$R_2 = 7{,}490 lb$$

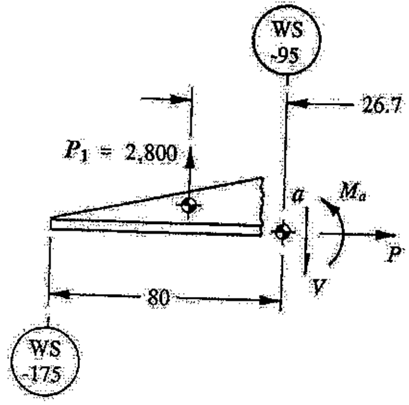
Hence, from $\sum F_y = 0$, R_1 is

$$-R_1 - 7{,}490 = -14{,}980$$

$$R_1 = 7{,}490 lb$$

Now, with a completely balanced structure, the internal forces V and M at point a and b are computed, and their values plotted on shear and bending moments diagrams, respectively.

Internal loads at point a,

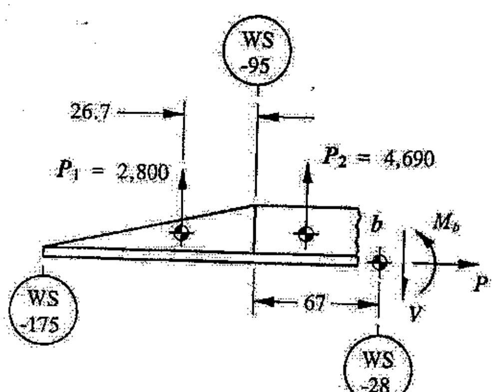


$\sum F_x = 0 \qquad P = 0$

$\sum F_y = 0 \qquad 2{,}800 - V = 0 \qquad V = 2{,}800 lb$

$\sum M_a = 0 \qquad 2{,}800(26.7) - M_a = 0 \qquad M_a = 74{,}760 lb - in$

Internal loads at point b,

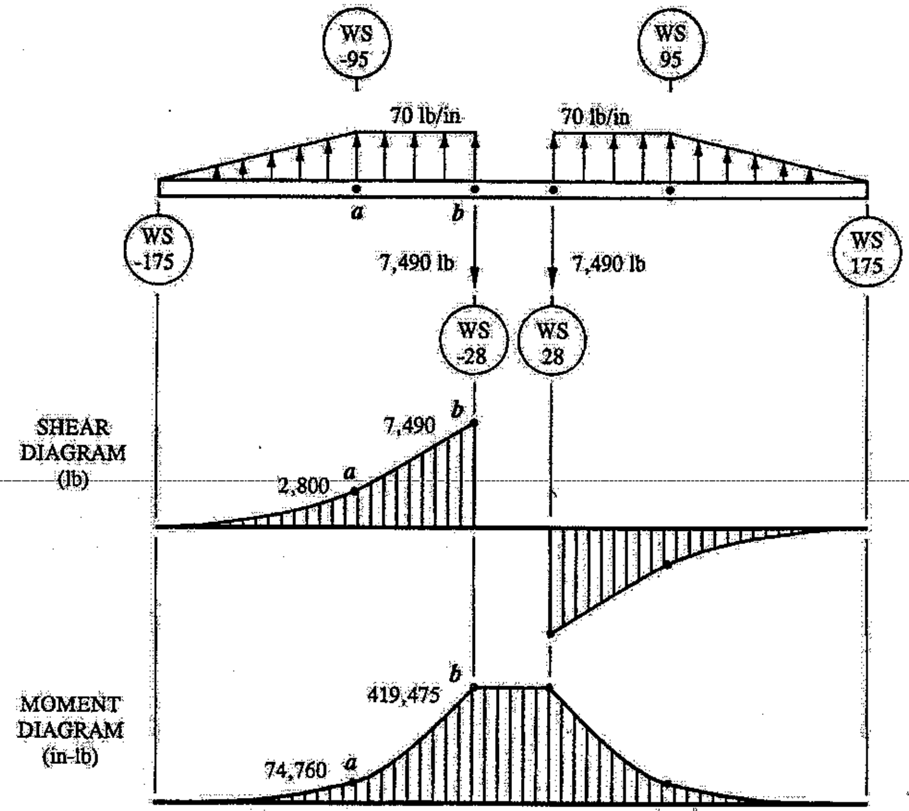


$\sum F_x = 0 \qquad P = 0$

$\sum F_y = 0 \qquad 2{,}800 + 4{,}690 - V = 0 \qquad V = 7{,}490 lb$

$\sum M_b = 0 \qquad 2{,}800(93.7) + 4{,}690(33.5) - M_b = 0 \qquad M_b = 419{,}475 lb - in$

Finally,


WS -95
WS 95
70 lb/in
70 lb/in
a
b
WS -175
7,490 lb
7,490 lb
WS 175
WS -28
WS 28
SHEAR DIAGRAM (lb)
7,490
b
2,800
a
MOMENT DIAGRAM (in-lb)
419,475
b
74,760
a