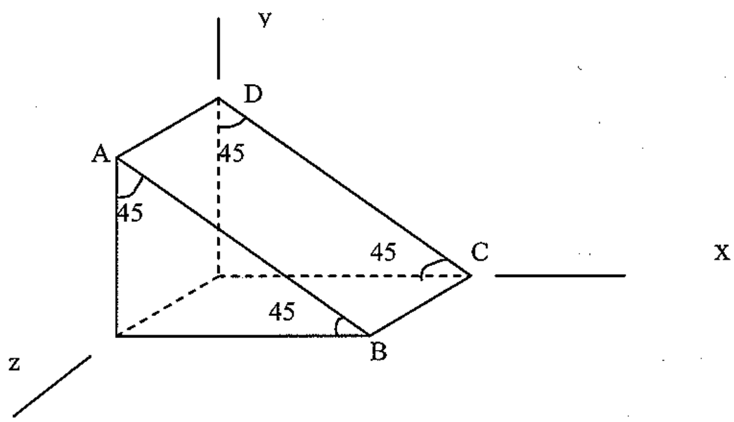
MAE3315
Homework No.2 (SOLUTION)

Problem 2-1
A state of stress is given at a point on the vertical fin of an aircraft as

$$[\sigma]=\begin{bmatrix}\sigma_x & \tau_{xy}\\ \tau_{xy} & \sigma_y\end{bmatrix}=\begin{bmatrix}30 & 4\\ 4 & 10\end{bmatrix}\ \text{Ksi}$$

Find

a) The normal and shear stress components on the plane ABCD.
b) The corresponding stress components if the original plane is rotated 45° about the z-axis.



Solution:

a)

First, we need the projection of {n} on {x} and {y}.

$$n_x = Cos(\theta) = Cos(45) = \frac{1}{\sqrt{2}} \quad \text{and} \quad n_y = Sin(\theta) = Sin(45) = \frac{1}{\sqrt{2}}$$

Therefore, the vector {n} becomes,

$$\{n\} = \begin{Bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{Bmatrix}$$

On the other hand, $\{t\} = [\sigma]\cdot\{n\}$,

$$\begin{Bmatrix}t_x\\ t_y\end{Bmatrix} = \begin{bmatrix}30 & 4\\ 4 & 10\end{bmatrix}\cdot\begin{Bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{Bmatrix} = \begin{Bmatrix}34/\sqrt{2}\\ 14/\sqrt{2}\end{Bmatrix}$$

Finally, $\sigma_n = \vec{t} \cdot \vec{n}$,

$$\sigma_n = \vec{t} \cdot \vec{n} = t_x n_x + t_y n_y = \left(\frac{34}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{14}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{34}{2} + \frac{14}{2} = 24Ksi$$

Now,

$$\sigma_n^{\ 2} + S^2 = \|t\|^2$$

$$S = \sqrt{\|t\|^2 - \sigma_n^{\ 2}}$$

Where,

$$\|t\| = \sqrt{t_x^2 + t_y^2} = \sqrt{\left(\frac{34}{\sqrt{2}}\right)^2 + \left(\frac{14}{\sqrt{2}}\right)^2} = \sqrt{578 + 98} = 26Ksi$$

$$S = \sqrt{(26)^2 - (24)^2}$$

$$S = 10Ksi$$

With,

$$\vec{\tau} = \frac{\vec{t} - \sigma_n \vec{n}}{S}$$

$$\vec{\tau} = \frac{\begin{Bmatrix} 34/\sqrt{2} \\ 14/\sqrt{2} \end{Bmatrix} - 24 \begin{Bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{Bmatrix}}{10} = \begin{Bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{Bmatrix}$$

This means that the direction of the Shear Stress is negative in our convention. Therefore, S= -10ksi. <u>**There is another way to solve for the Shear Stress,**</u>

To calculate the components of the vector {τ},

$$\tau_x = -Sin(\theta) = -Sin(45) = -\frac{1}{\sqrt{2}} \quad \text{and} \quad \tau_y = Cos(\theta) = Cos(45) = \frac{1}{\sqrt{2}}$$

As a result, the vector {τ} becomes,

$$\{\tau\} = \begin{Bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{Bmatrix}$$

And then, $S = \vec{t} \cdot \vec{\tau}$,

$$S = \vec{t} \cdot \vec{\tau} = t_x \tau_x + t_y \tau_y = \left(\frac{34}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right) + \left(\frac{14}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{-34}{2} + \frac{14}{2} = -10Ksi$$

b)

$$\left[\sigma_{x'-y'}\right] = \left[T_\sigma(\theta)\right]\left[\sigma_{x-y}\right]$$

$$\begin{bmatrix} \sigma_{x'} \\ \sigma_{y'} \\ \tau_{x'y'} \end{bmatrix} = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix} \cdot \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \quad \text{where,} \quad \begin{cases} m = \cos\theta \\ n = \sin\theta \end{cases}$$

$$\begin{bmatrix} \sigma_{x'} \\ \sigma_{y'} \\ \tau_{x'y'} \end{bmatrix} = \begin{bmatrix} (1/\sqrt{2})^2 & (1/\sqrt{2})^2 & 2(1/\sqrt{2})(1/\sqrt{2}) \\ (1/\sqrt{2})^2 & (1/\sqrt{2})^2 & -2(1/\sqrt{2})(1/\sqrt{2}) \\ -(1/\sqrt{2})(1/\sqrt{2}) & (1/\sqrt{2})(1/\sqrt{2}) & (1/\sqrt{2})^2 - (1/\sqrt{2})^2 \end{bmatrix} \cdot \begin{bmatrix} 30 \\ 10 \\ 4 \end{bmatrix} Ksi$$

$$\begin{bmatrix} \sigma_{x'} \\ \sigma_{y'} \\ \tau_{x'y'} \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & -1 \\ -1/2 & 1/2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 30 \\ 10 \\ 4 \end{bmatrix} Ksi$$

$$\begin{bmatrix} \sigma_{x'} \\ \sigma_{y'} \\ \tau_{x'y'} \end{bmatrix} = \begin{bmatrix} 24 \\ 16 \\ -10 \end{bmatrix} Ksi$$

Same results as part a, as expected!

Problem 2-2

A stress state at a point **P** on a plane is given as below.

$$[\sigma]=\begin{bmatrix}1 & -3 & 5\\ -3 & 2 & 2\\ 5 & 2 & 3\end{bmatrix}\cdot x10^3\,Psi$$

Consider a plane, $4x+3y+6z=12$ through the point P.

a) Find the normal component of the traction σ_n and its tangential component S acting at that point P on this plane.
b) Find the direction cosines of σ_n and S.
c) Find the magnitude of the traction vector **t** and its angle with the normal vector **n**

Solution:

a)

First, we need to find the directional cosines of that plane $4x+3y+6z=12$. From this equation is easy to compute that: a=3, b=4, and c=2.

$$Cos\alpha=\frac{bc}{\sqrt{(bc)^2+(ac)^2+(ab)^2}}=\frac{(4)(2)}{\sqrt{((4)(2))^2+((3)(2))^2+((3)(4))^2}}=0.5121=l$$

$$Cos\beta=\frac{ac}{\sqrt{(bc)^2+(ac)^2+(ab)^2}}=\frac{(3)(2)}{\sqrt{((4)(2))^2+((3)(2))^2+((3)(4))^2}}=0.3841=m$$

$$Cos\gamma=\frac{ab}{\sqrt{(bc)^2+(ac)^2+(ab)^2}}=\frac{(3)(4)}{\sqrt{((4)(2))^2+((3)(2))^2+((3)(4))^2}}=0.7682=n$$

$$\vec{n}=\begin{Bmatrix}0.5121\\0.3841\\0.7682\end{Bmatrix}$$

As we know, $\{t\}=[\sigma]\cdot\{n\}$,

$$\{t\}=\begin{bmatrix}1 & -3 & 5\\ -3 & 2 & 2\\ 5 & 2 & 3\end{bmatrix}Ksi\begin{Bmatrix}0.5121\\0.3841\\0.7682\end{Bmatrix}=\begin{Bmatrix}3.2009\\0.7682\\5.6336\end{Bmatrix}Ksi$$

$$\|\vec{t}\| = \sqrt{(3.2009)^2 + (0.7682)^2 + (5.6336)^2} = 6.5249\ Ksi$$

Finally, $\sigma_n = \vec{t} \cdot \vec{n}$,

$$\sigma_n = \vec{t} \cdot \vec{n} = t_x n_x + t_y n_y + t_z n_z = (3.2009)(0.5121) + (0.7682)(0.3841) + (5.6336)(0.7682) = 6.2623 Ksi$$

And,

$$\sigma_n{}^2 + S^2 = \|\vec{t}\|^2$$

$$S = \sqrt{\|\vec{t}\|^2 - \sigma_n{}^2}$$

$$S = \sqrt{(6.5249)^2 - (6.2623)^2} = 1.8323 Ksi$$

b)

The directional cosines of σ_n is naturally {n},

$$\vec{n} = \begin{Bmatrix} 0.5121 \\ 0.3841 \\ 0.7682 \end{Bmatrix}$$

And the directional cosines of $\vec{\tau}$ are,

$$\vec{\tau} = \frac{\vec{t} - \sigma_n \vec{n}}{S}$$

$$\vec{\tau} = \frac{\begin{Bmatrix} 3.2009 \\ 0.7682 \\ 5.6336 \end{Bmatrix} - 6.2623 \begin{Bmatrix} 0.5121 \\ 0.3841 \\ 0.7682 \end{Bmatrix}}{1.8323} = \begin{Bmatrix} -0.0034 \\ -0.8935 \\ 0.4490 \end{Bmatrix}$$

c)

The magnitude of {t} was already calculated,

$$\|\vec{t}\| = 6.5249\ Ksi$$

To calculate the angle, we know that,

$$\tan\theta = \frac{S}{\sigma_n}$$

$$\theta = ArcTan\left(\frac{S}{\sigma_n}\right) = 0.2847 radians = 16.3^\circ$$

Another way to solve it,

$$[\vec{t}]\cdot[\hat{n}] = \|\vec{t}\|\|\hat{n}\|\cos\theta$$

$$\theta = ArcCos\left(\frac{[\vec{t}]\cdot[\hat{n}]}{\|\vec{t}\|}\right)$$

$$\theta = ArcCos\left(\frac{\sigma_n}{\|\vec{t}\|}\right)$$

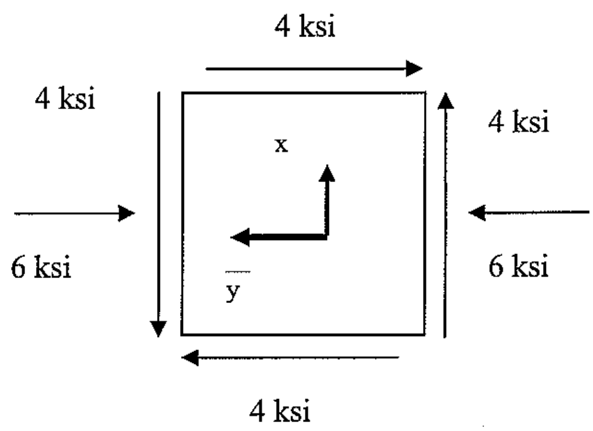
$$\theta = ArcCos\left(\frac{6.2623 MPa}{6.5249\ MPa}\right)$$

$$\theta = 16.3^\circ$$

Same answer as expected!

Problem 2-3

A stress element has the state of stress as shown below:
Find the principal stresses and their corresponding directions.
Note: Use the method given in the class only.



Solution:

$$[\sigma]=\begin{bmatrix} 0 & -4 \\ -4 & -6 \end{bmatrix} ksi$$

$$\begin{vmatrix} 0-\sigma_P & -4 \\ -4 & -6-\sigma_P \end{vmatrix} = (-\sigma_P)(-6-\sigma_P)-16=0$$

$$(-\sigma_P)(-6-\sigma_P)-16=0$$

$$\sigma_P^2+6\sigma_P-16=0$$

Therefore, the principal stresses are:

$$\begin{cases} \sigma_1 = 2Ksi \\ \sigma_2 = 0Ksi \\ \sigma_3 = -8Ksi \end{cases}$$

For $\sigma_1 = 2Ksi$:

$$\begin{bmatrix} 0-2 & -4 \\ -4 & -6-2 \end{bmatrix} \cdot \begin{Bmatrix} n_x^{(1)} \\ n_y^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{cases} -2n_x^{(1)} - 4n_y^{(1)} = 0 \\ \left(n_x^{(1)}\right)^2 + \left(n_y^{(1)}\right)^2 = 1 \end{cases}$$

$$\begin{Bmatrix} n_x^{(1)} \\ n_y^{(1)} \end{Bmatrix} = \begin{Bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{Bmatrix}$$

For $\sigma_3 = -8Ksi$:

$$\begin{bmatrix} 0-(-8) & -4 \\ -4 & -6-(-8) \end{bmatrix} \cdot \begin{Bmatrix} n_x^{(3)} \\ n_y^{(3)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{cases} 8n_x^{(3)} - 4n_y^{(3)} = 0 \\ \left(n_x^{(3)}\right)^2 + \left(n_y^{(3)}\right)^2 = 1 \end{cases}$$

$$\begin{Bmatrix} n_x^{(3)} \\ n_y^{(3)} \end{Bmatrix} = \begin{Bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{Bmatrix}$$