MAE 3315
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SOLUTION HW #3

Problem #1:

Part A) strain energy:

First, we need to realize that the torque acting on the left rod is 3T and on the right on is 1T.

From torsion theory we know that,

$$\tau_{r\theta} = \frac{T \cdot r}{J} \qquad \text{where} \quad J = \frac{\pi \cdot r^4}{2}$$

In addition, from Hook's Law, we know that,

$$\gamma_{r\theta} = \frac{1}{G}\tau_{r\theta}$$

Therefore, the strain energy equation can be rewritten as,

$$\Delta U = \frac{1}{2}(\tau_{r\theta}\gamma_{r\theta})\Delta V = \frac{1}{2}\left(\frac{1}{G}\tau_{r\theta}{}^2\right)\Delta V$$

To calculate the volume, we just substitutes, $\Delta V = rdrd\theta dl$. Finally, we can calculate the strain energy applying all these equations and integrating over the whole volume for each bar.

$$\Delta U = \iiint_V \frac{1}{2}\left(\frac{1}{G}\tau_{r\theta}{}^2\right) rdrd\theta dl$$

$$\Delta U = \frac{1}{2G}\iiint_V \left(\frac{T^2 \cdot r^2}{J^2}\right) rdrd\theta dl$$

$$\Delta U = \frac{1}{2G}\iiint_V \left(\frac{T^2 \cdot r^2}{\frac{\pi^2 \cdot R^8}{4}}\right) rdrd\theta dl$$

$$\Delta U = \frac{2T^2}{G\pi^2 R^8} \iiint_V r^3 \, dr\, d\theta\, dl$$

$$\Delta U = \frac{2T^2}{G\pi^2 R^8} \int_0^L \int_0^{2\pi} \int_0^R r^3 \, dr\, d\theta\, dl$$

$$\Delta U = \frac{2T^2}{G\pi^2 R^8} \left[\frac{r^4}{4}\right]_0^R \int_0^L \int_0^{2\pi} d\theta\, dl$$

$$\Delta U = \frac{T^2}{2G\pi^2 R^4} \left[\theta\right]_0^{2\pi} \int_0^L dl$$

$$\Delta U = \frac{2\pi T^2}{2G\pi^2 R^4} \left[l\right]_0^L$$

$$\Delta U = \frac{T^2 L}{G\pi R^4} = \frac{1}{2} \cdot \frac{T^2 L}{GJ}$$

<u>For the left rod:</u>

$$\Delta U_L = \frac{(3T)^2 (1.2a)}{G\pi (0.7d)^4}$$

$$\Delta U_L = 14.3180 \frac{T^2 a}{Gd^4}$$

<u>For the right rod:</u>

$$\Delta U_R = \frac{(T)^2 (a)}{G\pi (0.5d)^4}$$

$$\Delta U_R = 5.0930 \frac{T^2 a}{Gd^4}$$

<u>For both:</u>

$$\Delta U_{TOTAL} = \Delta U_L + \Delta U_R$$

$$\Delta U_{TOTAL} = 19.411 \frac{T^2 a}{G d^4}$$

Part B) substituting the values:

$$\Delta U_{TOTAL} = 19.411 \frac{(1400N - m)^2 (0.5m)}{(42x10^9\, Pa)(0.02m)^4}$$

$$\Delta U_{TOTAL} = 2831J$$

Problem #2:

We need to use these equations for each case in order to solve the problem,

$$\begin{cases} \varepsilon_x = \dfrac{\sigma_x}{E} - \nu\dfrac{\sigma_y}{E} - \nu\dfrac{\sigma_z}{E} & \text{Equation \#1} \\ \varepsilon_y = -\nu\dfrac{\sigma_x}{E} + \dfrac{\sigma_y}{E} - \nu\dfrac{\sigma_z}{E} & \text{Equation \#2} \\ \varepsilon_z = -\nu\dfrac{\sigma_x}{E} - \nu\dfrac{\sigma_y}{E} + \dfrac{\sigma_z}{E} & \text{Equation \#3} \end{cases}$$

Now for Plane Strain, $\varepsilon_z = 0$. In addition, since the edge is constrained, $\varepsilon_y = 0$. In consequence,

$$\begin{cases} \varepsilon_x = \dfrac{\sigma_x}{E} - \nu\dfrac{\sigma_y}{E} - \nu\dfrac{\sigma_z}{E} \\ \cancelto{0}{\varepsilon_y} = -\nu\dfrac{\sigma_x}{E} + \dfrac{\sigma_y}{E} - \nu\dfrac{\sigma_z}{E} \\ \cancelto{0}{\varepsilon_z} = -\nu\dfrac{\sigma_x}{E} - \nu\dfrac{\sigma_y}{E} + \dfrac{\sigma_z}{E} \end{cases}$$

$$\begin{cases} \varepsilon_x = \dfrac{\sigma_x}{E} - \nu\dfrac{\sigma_y}{E} - \nu\dfrac{\sigma_z}{E} & \\ \nu\sigma_z = \sigma_y - \nu\sigma_x & \text{Equation \#4} \\ \sigma_z = \nu\sigma_x + \nu\sigma_y & \text{Equation \#5} \end{cases}$$

From equations 4 and 5 solve σ_y and σ_z in terms of σ_x. Multiplying equation 4 by the Poisson's ratio and subtracting it from equation 5, (⑤$-\nu\cdot$④) we get,

$$(\sigma_y - \nu\sigma_x) - \nu^2(\sigma_x + \sigma_y) = 0$$

or

$$\sigma_y = \frac{\nu(1+\nu)\sigma_x}{1-\nu^2} \quad \rightarrow \quad \boxed{\sigma_y = \frac{\nu\sigma_x}{1-\nu}} \quad \text{Equation \#6}$$

Now from equations 4 and 6, we get,

$$\nu\sigma_z = \frac{\nu\sigma_x}{1-\nu} - \nu\sigma_x$$

$$\sigma_z = \sigma_x\left[\frac{1}{1-\nu} - 1\right]$$

$$\boxed{\sigma_z = \frac{\nu\sigma_x}{1-\nu}}$$

Equation #7

Substituting equations 6 and 7 into 1, we have,

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{1}{E}\left(\frac{\nu\sigma_x}{1-\nu}\right) - \nu\frac{1}{E}\left(\frac{\nu\sigma_x}{1-\nu}\right)$$

$$\varepsilon_x = \frac{\sigma_x}{E}\left[1 - \frac{\nu^2}{1-\nu} - \frac{\nu^2}{1-\nu}\right]$$

$$\varepsilon_x = \frac{\sigma_x}{E}\left[\frac{1-\nu-2\nu^2}{1-\nu}\right]$$

$$\sigma_x = \left(E\frac{1-\nu}{1-\nu-2\nu^2}\right)\varepsilon_x$$

Comparing with $\sigma_x = k\varepsilon_x$, we get that,

$$k = E\frac{1-\nu}{1-\nu-2\nu^2}$$

$$k = \left(70x10^9\, Pa\right)\frac{1-0.3}{1-0.3-2(0.3)^2}$$

$$\boxed{k = 94.23x10^9\, Pa}$$

For <u>Plane Stress</u>, $\sigma_z = 0$. Once again, since the edge is constrained, $\varepsilon_y = 0$. Therefore, equations 1, 2, and 3 becomes,

$$\begin{cases} \varepsilon_x = \dfrac{\sigma_x}{E} - \nu\dfrac{\sigma_y}{E} - \nu\cancelto{0}{\dfrac{\sigma_z}{E}} \\ \cancelto{0}{\varepsilon_y} = -\nu\dfrac{\sigma_x}{E} + \dfrac{\sigma_y}{E} - \nu\cancelto{0}{\dfrac{\sigma_z}{E}} \\ \varepsilon_z = -\nu\dfrac{\sigma_x}{E} - \nu\dfrac{\sigma_y}{E} + \cancelto{0}{\dfrac{\sigma_z}{E}} \end{cases}$$

$$\begin{cases} \varepsilon_x = \dfrac{\sigma_x}{E} - \nu\dfrac{\sigma_y}{E} & \text{Equation \#9} \\ \sigma_y = \nu\sigma_x & \text{Equation \#10} \\ \varepsilon_z = -\nu\dfrac{\sigma_x}{E} - \nu\dfrac{\sigma_y}{E} & \text{Equation \#11} \end{cases}$$

Substituting equation 10 into 9, we get,

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{(\nu\sigma_x)}{E}$$

$$\varepsilon_x = \frac{\sigma_x}{E}\left[1-\nu^2\right]$$

$$\sigma_x = \frac{E}{1-\nu^2}\varepsilon_x$$

Therefore,

$$k = \frac{E}{1-\nu^2}$$

$$k = \frac{70x10^9\,Pa}{1-(0.3)^2}$$

$$\boxed{k = 76.92x10^9\,Pa}$$

Problem #3:

For this problem we need the Hook's Law and Poisson's ratio equations,

$$\begin{cases} \varepsilon_x = \dfrac{\sigma_x}{E} - \nu\dfrac{\sigma_y}{E} - \nu\dfrac{\sigma_z}{E} \\ \varepsilon_y = -\nu\dfrac{\sigma_x}{E} + \dfrac{\sigma_y}{E} - \nu\dfrac{\sigma_z}{E} \\ \varepsilon_z = -\nu\dfrac{\sigma_x}{E} - \nu\dfrac{\sigma_y}{E} + \dfrac{\sigma_z}{E} \end{cases}$$

The longitudinal strain needs to be 1%, $\varepsilon_x = 0.01$. In addition, since the lateral surface of the bar is constrained, $\varepsilon_y = 0$ and $\varepsilon_z = 0$. As a result,

$$\begin{cases} 0.01 = \dfrac{\sigma_x}{E} - \nu\dfrac{\sigma_y}{E} - \nu\dfrac{\sigma_z}{E} \\ 0 = -\nu\sigma_x + \sigma_y - \nu\sigma_z \\ 0 = -\nu\sigma_x - \nu\sigma_y + \sigma_z \end{cases}$$

From these last 2 equations we can conclude that $\sigma_y = \sigma_z$. Substituting back we get,

$$0 = -\nu\sigma_x + \sigma_y(1-\nu)$$

$$\sigma_y = \frac{\nu}{(1-\nu)}\sigma_x$$

Now substituting back in the first equation, we get,

$$0.01 = \frac{\sigma_x}{E} - \nu\frac{1}{E}\frac{\nu}{(1-\nu)}\sigma_x - \nu\frac{1}{E}\frac{\nu}{(1-\nu)}\sigma_x$$

$$0.01 = \frac{\sigma_x}{E}\left[1 - \frac{\nu^2}{1-\nu} - \frac{\nu^2}{1-\nu}\right]$$

$$0.01 = \frac{\sigma_x}{E}\left(\frac{1-\nu-2\nu^2}{1-\nu}\right)$$

$$\sigma_x = (0.01)E\left(\frac{1-\nu}{1-\nu-2\nu^2}\right)$$

$$\sigma_x = (0.01)(10.5x10^6\,psi)\left(\frac{1-0.33}{1-0.33-2(0.33)^2}\right)$$

$$\sigma_x = 155{,}570\,psi$$

However, we want to know the load not the stress. Therefore, we need to multiply it by the area.

$$P = \sigma_x \cdot Area$$

$$P = (155{,}570\,psi)\cdot(1in^2) = \boxed{155{,}570\,lb}$$

For a bar under simple tension, $\sigma_y = \sigma_z = 0$, and the longitudinal strain remains $\varepsilon_x = 0.01$. Therefore,

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\cancelto{0}{\frac{\sigma_y}{E}} - \nu\cancelto{0}{\frac{\sigma_z}{E}}$$

$$0.01 = \frac{\sigma_x}{E}$$

$$\sigma_x = 0.01(10.5x10^6\,psi) = 105{,}00\,psi$$

$$P = (105{,}00\,psi)\cdot(1in^2) = \boxed{105{,}000\,lb}$$

The difference between these 2 cases is,

$$Diff\% = \left|\frac{155{,}570-105{,}000}{105{,}000}\right| x100\% = \boxed{48.2\%}$$

Problem #4:

Part A) principal stresses:

First, we need to get σ_x, σ_y, and τ_{xy}.

$$\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} = \begin{bmatrix} 10 & 12 \\ 12 & 0 \end{bmatrix} Ksi$$

To calculate the principal stresses, we need to,

$$\begin{vmatrix} 10-\sigma & 12 \\ 12 & 0-\sigma \end{vmatrix} = 0$$

$$(10-\sigma)(-\sigma)-144=0$$

$$\sigma^2 - 10\sigma - 144 = 0$$

Therefore, the principal stresses are:

$$\boxed{\begin{cases} \sigma_1 = 18Ksi \\ \sigma_2 = -8Ksi \end{cases}}$$

Part B) principal directions:

For $\sigma_1 = 18Ksi$:

$$\begin{bmatrix} -8 & 12 \\ 12 & -18 \end{bmatrix} \cdot \begin{Bmatrix} n_x^{(1)} \\ n_y^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{cases} -8n_x^{(1)} + 12n_y^{(1)} = 0 \\ 12n_x^{(1)} - 18n_y^{(1)} = 0 \end{cases}$$

However, these 2 equations are nor independent from each other. Therefore, I need 1 more equation which is $\left(n_x^{(1)}\right)^2 + \left(n_y^{(1)}\right)^2 = 1$.

$$\boxed{\begin{Bmatrix} n_x^{(1)} \\ n_y^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0.832 \\ 0.555 \end{Bmatrix}}$$

For $\sigma_2 = -8Ksi$:

$$\begin{bmatrix} 18 & 12 \\ 12 & 8 \end{bmatrix} \cdot \begin{Bmatrix} n_x^{(2)} \\ n_y^{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{cases} 18n_x^{(2)} + 12n_y^{(2)} = 0 \\ 12n_x^{(2)} + 8n_y^{(2)} = 0 \end{cases}$$

Once again, I need 1 to use the following equation $\left(n_x^{(2)}\right)^2 + \left(n_y^{(2)}\right)^2 = 1$.

$$\boxed{\begin{Bmatrix} n_x^{(2)} \\ n_y^{(2)} \end{Bmatrix} = \begin{Bmatrix} -0.555 \\ 0.832 \end{Bmatrix}}$$

Part C) principal strains:

The principal strains are produced by the principal stresses. Therefore,

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} \\ -\frac{\nu}{E} & \frac{1}{E} \end{bmatrix} \cdot \begin{Bmatrix} \sigma_1 \\ \sigma_2 \end{Bmatrix}$$

or

$$\begin{cases} \varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu \cdot \sigma_2\right] \\ \varepsilon_2 = \frac{1}{E}\left[\sigma_2 - \nu \cdot \sigma_1\right] \end{cases}$$

$$\begin{cases} \varepsilon_1 = \frac{1}{10x10^6\, psi}\left[(18{,}000\, psi) - (0.3)\cdot(-8{,}000\, psi)\right] \\ \varepsilon_2 = \frac{1}{10x10^6\, psi}\left[(-8{,}000\, psi) - (0.3)\cdot(18{,}000\, psi)\right] \end{cases}$$

$$\boxed{\begin{matrix} \varepsilon_1 = 2040\mu\varepsilon \\ \varepsilon_2 = -1340\mu\varepsilon \end{matrix}}$$

Part D) maximum shear stress:

To calculate the maximum shear stress, we only need to take derivative of the following equation with respect to θ,

$$\tau_{xy}' = \frac{1}{2}(\sigma_y - \sigma_x)Sin(2\theta) + \tau_{xy}Cos(2\theta)$$

Therefore, to calculate the maximum of this equation,

$$\frac{\partial}{\partial\theta}\tau_{xy}' = 0$$

$$\frac{\partial}{\partial\theta}\left[\frac{1}{2}(\sigma_y - \sigma_x)Sin(2\theta_s) + \tau_{xy}Cos(2\theta_s)\right] = 0$$

$$\left[(\sigma_y - \sigma_x)Cos(2\theta_s) - 2\tau_{xy}Sin(2\theta_s)\right] = 0$$

$$2\tau_{xy}Sin(2\theta_s) = (\sigma_y - \sigma_x)Cos(2\theta_s)$$

$$\frac{Sin(2\theta_s)}{Cos(2\theta_s)} = \frac{(\sigma_y - \sigma_x)}{2\tau_{xy}}$$

$$Tan(2\theta_s) = \frac{(\sigma_y - \sigma_x)}{2\tau_{xy}}$$

$$\theta_s = \frac{1}{2}ArcTan\left(\frac{(\sigma_y - \sigma_x)}{2\tau_{xy}}\right)$$

Finally, substituting this back in the first equation, we get that,

$$\tau_{\max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \pm\frac{1}{2}(\sigma_1 - \sigma_2)$$

In our case,

$$\tau_{\max} = \pm\frac{1}{2}(18 - (-8)) = \boxed{\pm 13Ksi}$$

Part E) maximum shear plane:

From the section before we got that,

$$\theta_s = \frac{1}{2} ArcTan\left(\frac{(\sigma_y - \sigma_x)}{2\tau_{xy}}\right)$$

In our case,

$$\theta_s = \frac{1}{2} ArcTan\left(\frac{(0-10)}{2(12)}\right)$$

$$\theta_s = \frac{1}{2} ArcTan\left(\frac{(0-10)}{2(12)}\right)$$

$$\theta_s = \frac{1}{2} ArcTan\left(\frac{-10}{24}\right)$$

$$\boxed{\theta_s = -11.3^\circ}$$

To check my answer I know that the plane of maximum shearing stress is inclined at 45° with respect to the planes of principal stress. To check this, we use our previous result,

$$\begin{Bmatrix} n_x^{(1)} \\ n_y^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0.832 \\ 0.555 \end{Bmatrix}$$

And from there, we know that, $n_x^{(1)} = Cos(\theta_p)$. Therefore,

$$\theta_p = ArcCos(0.832) = 33.7^\circ$$

Finally, between θ_s and θ_p are indeed 45° as expected!!!

$$\theta_p - \theta_s = (33.7^\circ) - (-11.3^\circ) = 45^\circ$$