MAE3315

HW # 4. (SOLUTION) (REVISED '2')

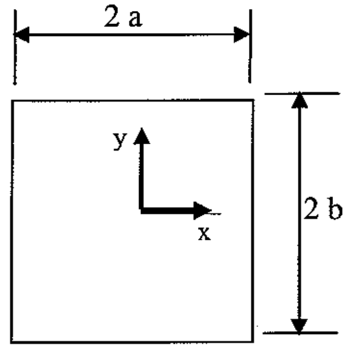
Problem 4-1

A rectangular cross-section with the edges, 2a *by 2b* is subjected to a torque, *T*.

a) What is Prandle stress equation, φ in term of *T, a* and *b?*

b) Determine the shear stress along the vertical and horizontal sides, respectively.

c) Are the shear stresses at the corner point continuous? Show why!



Solution:

a)

$$\phi = C(x-a)(x+a)(y-b)(y+b)$$

$$\phi = C(x^2-a^2)(y^2-b^2)$$

On the other hand,

$$T = 2\iint_A \phi dxdy$$

$$T = 2\int_{-b}^{b}\int_{-a}^{a} C(x^2-a^2)(y^2-b^2)dxdy$$

$$T = 2C\int_{-b}^{b}\left(\frac{x^3}{3}-a^2x\right)\Bigg|_{-a}^{a}(y^2-b^2)dy$$

$$T = 2C\int_{-b}^{b}\left(-\frac{4}{3}a^3\right)(y^2-b^2)dy$$

$$T = 2C\left(-\frac{4}{3}a^3\right)\left(\frac{y^3}{3} - b^2 y\right)\Bigg|_{-b}^{b}$$

$$T = 2C\left(-\frac{4}{3}a^3\right)\left(-\frac{4}{3}b^3\right)$$

$$T = \frac{32}{9}Ca^3b^3$$

$$C = \frac{9}{32}\frac{T}{a^3b^3}$$

Finally,

$$\boxed{\phi = \frac{9}{32}\frac{T}{a^3b^3}\left(x^2 - a^2\right)\left(y^2 - b^2\right)}$$

b)

Vertical side:

$$\tau_{xz} = \frac{\delta\phi}{\delta y} = \frac{\delta}{\delta y}\left[\frac{9}{32}\frac{T}{a^3b^3}\left(x^2 - a^2\right)\left(y^2 - b^2\right)\right] = \frac{9}{16}\frac{T}{a^3b^3}\left(x^2 - a^2\right)y$$

$$\tau_{yz} = -\frac{\delta\phi}{\delta x} = -\frac{\delta}{\delta x}\left[\frac{9}{32}\frac{T}{a^3b^3}\left(x^2 - a^2\right)\left(y^2 - b^2\right)\right] = -\frac{9}{16}\frac{T}{a^3b^3}\left(y^2 - b^2\right)x$$

Now substituting, x =± a,

$$\boxed{\tau_{xz}\big|_{x=\pm a} = 0}$$

$$\boxed{\tau_{yz}\big|_{x=\pm a} = \mp\frac{9}{16}\frac{T}{a^2b^3}\left(y^2 - b^2\right)}$$

Horizontal side:

$$\tau_{xz} = \frac{\delta\phi}{\delta y} = \frac{\delta}{\delta y}\left[\frac{9}{32}\frac{T}{a^3b^3}\left(x^2 - a^2\right)\left(y^2 - b^2\right)\right] = \frac{9}{16}\frac{T}{a^3b^3}\left(x^2 - a^2\right)y$$

$$\tau_{yz} = -\frac{\delta\phi}{\delta x} = -\frac{\delta}{\delta x}\left[\frac{9}{32}\frac{T}{a^3b^3}\left(x^2-a^2\right)\left(y^2-b^2\right)\right] = -\frac{9}{16}\frac{T}{a^3b^3}\left(y^2-b^2\right)x$$

Now substituting, y = $\pm$ b,

$$\boxed{\tau_{xz}\big|_{y=\pm b} = \pm\frac{9}{16}\frac{T}{a^3b^2}\left(x^2-a^2\right)}$$

$$\boxed{\tau_{yz}\big|_{y=\pm b} = 0}$$

c)

Now, evaluating the stresses along the vertical and horizontal side in the right-upper corner,

For the vertical wall,

$$\tau_{xz}\big|_{x=a,y=b} = 0$$

$$\tau_{yz}\big|_{x=\pm a,y=\pm b} = \mp\frac{9}{16}\frac{T}{a^2b^3}\left(\left(\pm b\right)^2-b^2\right) = 0$$

For the horizontal wall,

$$\tau_{xz}\big|_{y=\pm b,x=\pm a} = \pm\frac{9}{16}\frac{T}{a^3b^2}\left(\left(\pm a\right)^2-a^2\right) = 0$$

$$\tau_{yz}\big|_{y=b,x=a} = 0$$

The stresses in both walls have the same value, so this shows that the shear stress is continuous at the corners.

Problem 4-2

Consider two bars, one having a circular section of radius b, the other an elliptical section with major semi-axis a, and minor semi-axis b. Determine

a) for equal angles of twist, which bar experiences the large shearing stress
b) for equal allowable shearing stresses, which bar resists a large torque.

Solution:

a)

For the circle cross-section,

$$\phi_C = C_C\left(x^2 + y^2 - b^2\right)$$

$$\nabla^2 \phi_C = -2G\theta$$

$$\frac{\delta^2}{\delta x^2}\left[C_C\left(x^2 + y^2 - b^2\right)\right] + \frac{\delta^2}{\delta y^2}\left[C_C\left(x^2 + y^2 - b^2\right)\right] = -2G\theta$$

$$2C_C + 2C_C = -2G\theta$$

$$C_C = -\frac{G\theta}{2}$$

Substituting back,

$$\phi_C = -\frac{G\theta}{2}\left(x^2 + y^2 - b^2\right)$$

On the other hand, for the elliptical cross-section,

$$\phi_E = C_E\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$

$$\nabla^2 \phi_E = -2G\theta$$

$$\frac{\delta^2}{\delta x^2}\left[C_E\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\right]+\frac{\delta^2}{\delta y^2}\left[C_E\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\right]=-2G\theta$$

$$\frac{2}{a^2}C_E+\frac{2}{b^2}C_E=-2G\theta$$

$$\frac{\left(b^2+a^2\right)}{a^2b^2}C_E=-G\theta$$

$$C_E=-G\theta\frac{a^2b^2}{b^2+a^2}$$

Substituting back,

$$\phi_E=-G\theta\frac{a^2b^2}{b^2+a^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$$

Now, as we know, the maximum stress on the ellipse happens at the top and bottom of it (x=0, and y=±b). Choosing the bottom of both cross sections to compare them,

$$\tau_{xz,C}=\frac{\delta\phi_C}{\delta y}=\frac{\delta}{\delta y}\left[-\frac{G\theta}{2}\left(x^2+y^2-b^2\right)\right]=-G\theta y$$

$$\tau_{yz,C}=-\frac{\delta\phi_C}{\delta x}=-\frac{\delta}{\delta x}\left[-\frac{G\theta}{2}\left(x^2+y^2-b^2\right)\right]=G\theta x$$

$$\tau_{xz,E}=\frac{\delta\phi_E}{\delta y}=\frac{\delta}{\delta y}\left[-G\theta\frac{a^2b^2}{b^2+a^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\right]=-G\theta\frac{a^2b^2}{b^2+a^2}\left(\frac{2y}{b^2}\right)=-2G\theta\frac{a^2}{b^2+a^2}y$$

$$\tau_{yz,E}=-\frac{\delta\phi_E}{\delta x}=-\frac{\delta}{\delta x}\left[-G\theta\frac{a^2b^2}{b^2+a^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\right]=G\theta\frac{a^2b^2}{b^2+a^2}\left(\frac{2x}{a^2}\right)=2G\theta\frac{b^2}{b^2+a^2}x$$

Evaluating at the bottom of the cross-section, (x = 0 and y = -b),

$$\tau_{xz,C} = -G\theta(-b) \quad \rightarrow \quad \frac{\tau_{xz,C}}{b} = G\theta$$

$$\tau_{yz,C} = 0$$

$$\tau_{xz,E} = -2G\theta\frac{a^2}{b^2+a^2}(-b) \quad \rightarrow \quad \frac{b^2+a^2}{2a^2b}\tau_{xz,E} = G\theta$$

$$\tau_{yz,E} = 0$$

Finally, because the problem requires that the angles of twist of both cases are the same, and it is the same material (so the shear moduli are the same too)

$$\frac{\tau_{xz,C}}{b} = \frac{b^2+a^2}{2a^2b}\tau_{xz,E}$$

$$\tau_{xz,C} = \frac{b^2+a^2}{2a^2}\tau_{xz,E}$$

But we know that a > b. Then, the term $\frac{b^2+a^2}{2a^2}$ is less than one, $\frac{b^2+a^2}{2a^2} < 1$. Therefore,

$$\boxed{\tau_{xz,C} < \tau_{xz,E}}$$

b)

From Supplement 3.3, for the ellipse,

$$\tau_{\max,E} = \frac{2T}{\pi a b^2}$$

On the other hand, for the circular cross-section,

$$\tau_{\max,C} = \frac{Tb}{J}$$

For equal allowable shearing stresses,

$$\frac{2T_E}{\pi a b^2} = \tau_{\max,E} = \tau_{\max,C} = \frac{T_C b}{J}$$

$$\frac{2T_E}{\pi a b^2} = \frac{T_C b}{J}$$

$$\frac{2T_E}{\pi a b^2} = \frac{T_C b}{\frac{\pi}{2} b^4}$$

$$\frac{T_E}{a} = \frac{T_C}{b}$$

Once again we know that a > b; therefore,

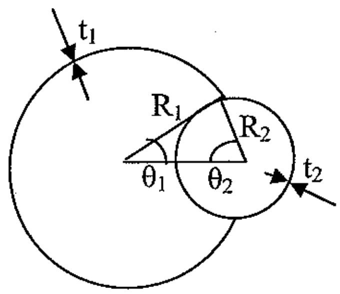
$$\boxed{T_E > T_C}$$

Problem 4-3

The figure shows a typical 2-cell tubular section as formed by a conventional foil shape, and having one interior web. An external torque, T of 83450 in-lb is applied. Determine the shear flow and shear stress of each segment.

R_1=5 in. and R_2=2 in

$$\begin{cases} \theta_1 = 30^\circ \\ \theta_2 = 60^\circ \\ t_1 = t_2 = 0.10in \end{cases}$$



Solution:

a)

Note: There was a typo in the problem statement, θ_1 is not 30°.

$$\frac{R_2}{Sin\theta_1} = \frac{R_1}{Sin60^\circ}$$

$$\theta_2 = 20.27^\circ$$

First, we need to find the length of each section.

$$S_1 = 2\pi R_1 - \frac{2(20.27^\circ)}{360^\circ} 2\pi R_1 = 27.88\text{in}$$

$$S_{12} = \frac{120^\circ}{360^\circ} 2\pi R_2 = \frac{2}{3}\pi R_2 = 4.19\text{in}$$

$$S_2 = 2\pi R_2 - \frac{120^\circ}{360^\circ} 2\pi R_2 = \frac{4}{3}\pi R_2 = 8.38in$$

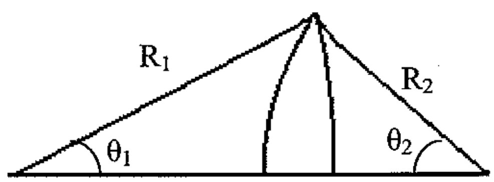
Then, we need to calculate the area bounded by the contours.

$$\overline{A}_{m2} = \pi R_2^2 = 12.57in^2$$

For the circle 1 the area is more complicated. First, the area of the whole circle is,

$$A_C = \pi R_1^2$$

Then, we need to subtract the portion taken out by circle 2.



$$A^* = \frac{1}{2}\theta_1 R_1^2 + \frac{1}{2}\theta_2 R_2^2 - \frac{1}{2}bh$$

$$A^* = \frac{1}{2}(0.3538rad)(5in)^2 + \frac{1}{2}\left(\frac{\pi}{3}\right)(2in)^2 - \frac{1}{2}(R_1 Cos\theta_1 + R_2 Cos\theta_2)(R_1 Sin\theta_1)$$

$$A^* = 1.588in^2$$

$$\overline{A}_{m1} = A_C - 2(A_1^*)$$

$$\overline{A}_{m1} = \pi R_1^2 - 2(1.588) = 75.36in^2$$

Now we know that,

$$T = 2\overline{A}_{m1}q_1 + 2\overline{A}_{m2}q_2$$

$$\frac{1}{\overline{A}_{m1}}\left[\frac{q_1 S_1}{t_1} + \frac{q_{12} S_{12}}{t_{12}}\right] = \frac{1}{\overline{A}_{m2}}\left[\frac{q_2 S_2}{t_2} - \frac{q_{12} S_{12}}{t_{12}}\right]$$

$$q_{12} = q_1 - q_2$$

Substituting those values in this system of 3 equations, and solving we get that,

$$q_1 = 494.46 lb/in$$

$$q_2 = 354.98 lb/in$$

$$q_{12} = 139.48 lb/in$$

In addition,

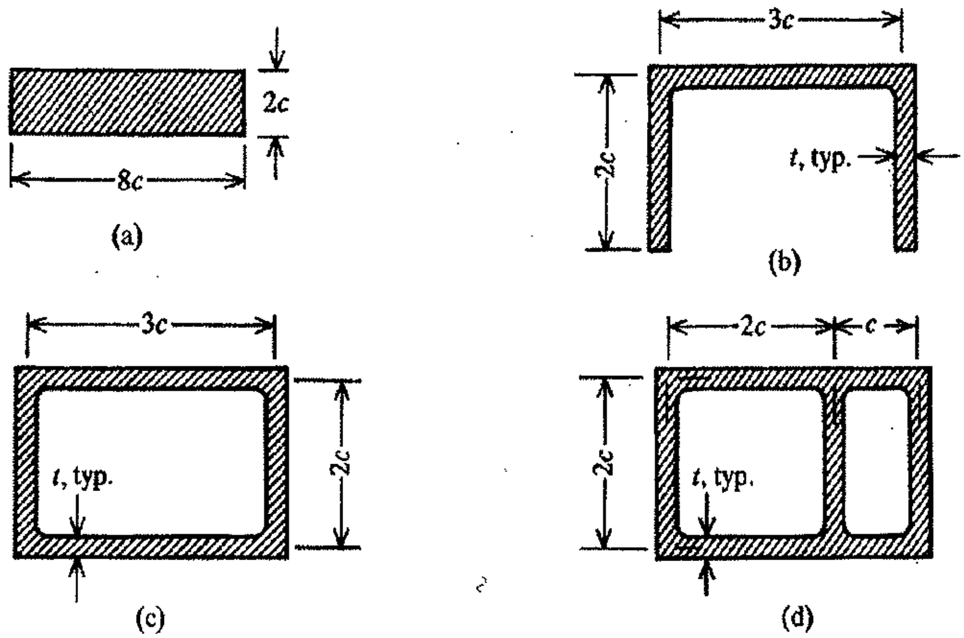
$$\tau_1 = \frac{q_1}{t_1} = \frac{494.46 lb/in}{0.10 in} = 4{,}944.6 psi$$

$$\tau_2 = \frac{q_2}{t_2} = \frac{354.98 lb/in}{0.10 in} = 3{,}549.8 psi$$

$$\tau_{12} = \frac{q_{12}}{t_{12}} = \frac{139.48 lb/in}{0.10 in} = 1{,}394.8 psi$$

Problem 4-4

The following beam cross-sections are made of the same material. All of the thin cross-sectional dimensions are centerline dimensions. Assume the dimension c to be much greater than the dimension t. Find the torsional constant, J for each case. For case a, J can be approximate as 0.28 (8c)(2c)^3. If the same torque is applied to the above sections, which one exhibits the largest twisting angle?



Solution:

a)

$$J_a = 0.28(8c)(2c)^3 = \boxed{18c^4}$$

b)

$$J_b = \frac{1}{3}\left[(2c)\cdot t^3 + (3c)\cdot t^3 + (2c)\cdot t^3\right] = \frac{1}{3}(7c)(t)^3 = \boxed{2.3ct^3}$$

c)

$$J_c = \frac{4\bar{A}^2}{\oint \frac{ds}{t}} = \frac{4(6c^2)^2}{\frac{10c}{t}} = \boxed{14.4c^3t}$$

d) Since there are two cells to this thin-walled closed structure, there is no simple formula for the torsion constant. The torsion constant is determinate by first calculating the two constant shear flows in the two-cell cross-section, and then calculating the twist per unit length. The torsion constant can be determined from the solution for the twist per unit length.

$$T = 2\overline{A}_1 q_1 + 2\overline{A}_2 q_2$$

$$T = 2\left(4c^2\right)q_1 + 2\left(2c^2\right)q_2$$

$$T = 8c^2 q_1 + 4c^2 q_2$$

On the other hand,

$$\frac{1}{\overline{A}_1}\left[\frac{q_1 S_1}{t_1} + \frac{q_{12} S_{12}}{t_{12}}\right] = \frac{1}{\overline{A}_2}\left[\frac{q_2 S_2}{t_2} - \frac{q_{12} S_{12}}{t_{12}}\right]$$

$$\frac{1}{\overline{A}_1}\left[q_1 S_1 + q_{12} S_{12}\right] = \frac{1}{\overline{A}_2}\left[q_2 S_2 - q_{12} S_{12}\right]$$

$$\frac{1}{4c^2}\left[q_1 6c + q_{12} 2c\right] = \frac{1}{2c^2}\left[q_2 4c - q_{12} 2c\right]$$

$$\frac{1}{4c^2}\left[q_1 6c + (q_1 - q_2)2c\right] = \frac{1}{2c^2}\left[q_2 4c - (q_1 - q_2)2c\right]$$

$$\left[q_1 6c + q_1 2c - q_2 2c\right] = 2\left[q_2 4c - q_1 2c + q_2 2c\right]$$

$$12 q_1 = 14 q_2$$

From these two equations, we get that,

$$q_1 = \frac{7}{80}\frac{T}{c^2} \quad \text{and} \quad q_2 = \frac{3}{40}\frac{T}{c^2}$$

Finally,

$$G\theta = \frac{1}{2\overline{A}_1}\left[\frac{q_1 S_1}{t_1} + \frac{q_{12} S_{12}}{t_{12}}\right]$$

$$G\theta = \frac{1}{2(4c^2)}\left[\frac{7}{80}\frac{T}{c^2}\frac{6c}{t} + \frac{1}{80}\frac{T}{c^2}\frac{2c}{t}\right]$$

$$G\theta = \frac{1}{8c^2}\left[\frac{11}{20}\frac{T}{ct}\right]$$

$$G\theta = \frac{11}{160}\frac{T}{c^3 t}$$

And,

$$J = \frac{T}{G\theta}$$

$$J = \frac{T}{\frac{11}{160}\frac{T}{c^3 t}}$$

$$\boxed{J_d = \frac{160}{11}c^3 t}$$

To answer the second question, for the same applied torque and material, the term T/G remains the same (it's a constant),

$$\frac{T}{G}\frac{1}{J} = \theta$$

$$\theta_a = \frac{T}{G}\frac{1}{18c^4}$$

$$\theta_b = \frac{T}{G}\frac{1}{2.3ct^3}$$

$$\theta_c = \frac{T}{G}\frac{1}{14.4c^3 t}$$

$$\theta_d = \frac{T}{G}\frac{1}{14.5c^3 t}$$

And since c >> t, θ_b should be the greatest. Therefore, cross-section "b" exhibits the largest twisting angle.

$$\theta_b > \theta_c \approx \theta_d > \theta_a$$