MAE 3315
Dr. Chan
SOLUTION HW #9

Problem #1:

Part a) Maximum principal stress criterion

First, we need to realize that at the surface of the shaft $\sigma_x = 0$, $\sigma_y = 0$, and $\tau_{xy} = \frac{T \cdot r}{J}$. Substituting the values in this equation,

$$\tau_{xy} = \frac{(6{,}000 lb - in)(r)}{\frac{\pi \cdot r^4}{2}} = \frac{2(6{,}000 lb - in)}{\pi \cdot r^3} = \frac{2(6{,}000 lb - in)}{\pi \cdot (d/2)^3} = \frac{16(6{,}000 lb - in)}{\pi \cdot (d)^3}$$

Therefore, for this problem, $[\sigma]$ becomes,

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} = \begin{bmatrix} 0 & \tau_{xy} \\ \tau_{xy} & 0 \end{bmatrix}$$

And to calculate the principal stresses,

$$\begin{vmatrix} 0-\lambda & \tau_{xy} \\ \tau_{xy} & 0-\lambda \end{vmatrix} = 0$$

$$(0-\lambda)(0-\lambda) - \tau_{xy}^2 = 0$$

$$\lambda^2 - \tau_{xy}^2 = 0$$

$$\lambda^2 = \tau_{xy}^2$$

$$\lambda = \pm\tau_{xy}$$

Finally,

$$\sigma_1 = +\tau_{xy} \quad \text{and} \quad \sigma_3 = -\tau_{xy}$$

NOTE: $\sigma_2 = 0$

Substituting,

$$\sigma_1 = \frac{16(6{,}000lb - in)}{\pi \cdot (d)^3} \quad \text{and} \quad \sigma_3 = -\frac{16(6{,}000lb - in)}{\pi \cdot (d)^3}$$

Now applying the Maximum principal stress criterion,

$$\sigma_1 > \sigma_0^T 0 \quad \text{or} \quad \sigma_3 < \sigma_0^C < 0 \quad \text{to fail}$$

NOTE: σ_0 is the yielding strength.

However, we don't want it to fail. Therefore,

$$\sigma_1 \le \sigma_0^T$$

In addition, we need to include the Safety Factor,

$$SF = \frac{\text{Failure Strength}}{\text{Actual Stress}}$$

$$SF = \frac{\sigma_0^T}{\sigma_1}$$

$$2 = \frac{60x10^3\, psi}{\frac{16(6{,}000lb - in)}{\pi \cdot (d)^3}}$$

$$60x10^3\, psi \cdot \pi \cdot (d)^3 = 2x16(6{,}000lb - in)$$

$$d = \sqrt[3]{\frac{2x16(6{,}000lb - in)}{\pi \cdot 60x10^3\, psi}}$$

$$\boxed{d = 1.006in}$$

In conclusion, the minimum required diameter for the shaft that will not give a yielding failure is $d = 1.0062in$ according to the Maximum principal stress criterion.

Part b) Maximum shear stress criterion

In this case, failure occurs when the maximum shear stress magnitude exceeds the maximum shear stress magnitude at yielding failure in a simple uniaxial load.

$$\left|\tau_{1,2}\right| > S \qquad \left|\tau_{2,3}\right| > S \qquad \left|\tau_{1,3}\right| > S \qquad \text{where } S = \frac{\sigma_0}{2},$$

In our case, we don't want it to fail,

$$\left|\tau_{1,3}\right| \le \frac{\sigma_0}{2}$$

To include the Safety Factor, we need to,

$$\frac{\sigma_1}{\sigma_3} = \frac{+\dfrac{16(6{,}000lb-in)}{\pi\cdot(d)^3}}{-\dfrac{16(6{,}000lb-in)}{\pi\cdot(d)^3}} = -1$$

From here, we get that,

$$\sigma_3^{allow} = (-1)\sigma_1^{allow}$$

Then from the Maximum shear stress criterion, we know that,

$$\frac{\sigma_1^{allow}}{\sigma_0} - \frac{\sigma_3^{allow}}{\sigma_0} = 1$$

Substituting in this equation,

$$\frac{\sigma_1^{allow}}{60Ksi} - \frac{-1\sigma_1^{allow}}{60Ksi} = 1$$

$$\frac{\sigma_1^{allow}}{30Ksi} = 1$$

$$\sigma_1^{allow} = 30Ksi$$

Again, $SF = \dfrac{\text{Failure Strength}}{\text{Actual Stress}}$

$$SF = \frac{\sigma_1^{allow}}{\sigma_1}$$

$$2 = \frac{30x10^3\,psi}{\dfrac{16(6{,}000lb-in)}{\pi\cdot(d)^3}}$$

$$30x10^3\, psi \cdot \pi \cdot (d)^3 = 2x16(6{,}000lb - in)$$

$$d = \sqrt[3]{\frac{2x16(6{,}000lb - in)}{\pi \cdot 30x10^3\, psi}}$$

$$\boxed{d = 1.268in}$$

Once again, the minimum required diameter for the shaft that will not give a yielding failure is $d = 1.268in$ according to the Maximum shear stress criterion.

Part c) Von-Mises failure criterion

$$\left(\sigma_1^{allow}\right)^2 + \left(\sigma_3^{allow}\right)^2 - \sigma_1^{allow} \cdot \sigma_3^{allow} = \sigma_0^2$$

However, from the previous section we already know that, $\sigma_3^{allow} = -\sigma_1^{allow}$. Substituting back,

$$\left(\sigma_1^{allow}\right)^2 + \left(-\sigma_1^{allow}\right)^2 - \sigma_1^{allow} \cdot \left(-\sigma_1^{allow}\right) = \sigma_0^2$$

$$2\left(\sigma_1^{allow}\right)^2 + \left(\sigma_1^{allow}\right)^2 = \sigma_0^2$$

$$3\left(\sigma_1^{allow}\right)^2 = \sigma_0^2$$

$$\sigma_1^{allow} = \frac{\sigma_0}{\sqrt{3}} = \frac{60Ksi}{\sqrt{3}} = 34.641Ksi$$

As usually, $SF = \frac{\text{Failure Strength}}{\text{Actual Stress}}$,

$$SF = \frac{\sigma_1^{allow}}{\sigma_1}$$

$$2 = \frac{34{,}641psi}{\frac{16(6{,}000lb - in)}{\pi \cdot (d)^3}}$$

$$34{,}641psi\pi \cdot (d)^3 = 2x16(6{,}000lb - in)$$

$$d = \sqrt[3]{\frac{2x16(6{,}000 lb - in)}{\pi \cdot 34{,}641 psi}}$$

$$\boxed{d = 1.208 in}$$

Therefore, the minimum required diameter for the shaft that will not give a yielding failure is $d = 1.208 in$ according to Von-Mises failure criterion.

Problem #2:

$$[\sigma]=\begin{bmatrix}\sigma_x & \tau_{xy}\\ \tau_{xy} & \sigma_y\end{bmatrix}=\begin{bmatrix}80 & 25\\ 25 & -40\end{bmatrix}MPa$$

First, we need to calculate the principal stresses,

$$\begin{vmatrix}80-\lambda & 25\\ 25 & -40-\lambda\end{vmatrix}=0$$

$$(80-\lambda)(-40-\lambda)-25^2=0$$

$$-3200-80\lambda+40\lambda+\lambda^2-625=0$$

$$\lambda^2-40\lambda-3825=0$$

$$\sigma_{1,3}=\frac{40\pm\sqrt{40^2-4(1)(-3825)}}{2(1)}$$

$$\sigma_{1,3}=\frac{40\pm 130}{2}$$

$$\sigma_{1,3}=20\pm 65$$

Finally,

$$\sigma_1=85MPa \quad \text{and} \quad \sigma_3=-45MPa$$

NOTE: $\sigma_2=0$

Part a) Maximum principal stress criterion

To calculate the Safety Factor,

$$SF=\frac{\text{Failure Strength}}{\text{Actual Stress}}$$

For tension,

$$SF=\frac{\sigma_0^T}{\sigma_1}$$

$$SF = \frac{250MPa}{85MPa} = 2.94$$

For compression,

$$SF = \frac{\sigma_0^C}{\sigma_3}$$

$$SF = \frac{-250MPa}{-45MPa} = 5.56$$

Therefore, the overall Safety Factor is the smallest of them SF=2.94.

Part b) Maximum shear stress criterion

In this case, we need to calculate σ_3^{allow},

$$\frac{\sigma_1}{\sigma_3} = \frac{85MPa}{-45Mpa} = -1.8889$$

From here, we get that,

$$\sigma_3^{allow} = -0.5294\sigma_1^{allow}$$

Then from the Maximum shear stress criterion, we know that,

$$\frac{\sigma_1^{allow}}{\sigma_0} - \frac{\sigma_3^{allow}}{\sigma_0} = 1$$

Substituting,

$$\frac{\sigma_1^{allow}}{250MPa} - \frac{-0.5294\sigma_1^{allow}}{250MPa} = 1$$

$$\frac{\sigma_1^{allow}}{250MPa} - \frac{-0.5294\sigma_1^{allow}}{250MPa} = 1$$

$$\frac{1.5294\sigma_1^{allow}}{250MPa} = 1$$

$$\sigma_1^{allow} = 163.5MPa$$

Again, $SF = \dfrac{\text{Failure Strength}}{\text{Actual Stress}}$

$$SF = \frac{\sigma_1^{allow}}{\sigma_1}$$

$$SF = \frac{163.5MPa}{85MPa} = 1.92$$

On the other hand,

$$SF = \frac{\sigma_3^{allow}}{\sigma_3} = \frac{-0.5294\sigma_1^{allow}}{-45MPa} = \frac{-0.5294(163.5MPa)}{-45MPa} = 1.92$$

NOTE: Please, note that both Safety Factors are exactly the same. This is expected!

Part c) Von-Mises failure criterion

$$\left(\sigma_1^{allow}\right)^2 + \left(\sigma_3^{allow}\right)^2 - \sigma_1^{allow} \cdot \sigma_3^{allow} = \sigma_0^2$$

Just as before,

$$\frac{\sigma_1}{\sigma_3} = \frac{85MPa}{-45Mpa} = -1.8889 \qquad \Rightarrow \qquad \sigma_3^{allow} = -0.5294\sigma_1^{allow}$$

Substituting back,

$$\left(\sigma_1^{allow}\right)^2 + \left(-0.5294\sigma_1^{allow}\right)^2 - \sigma_1^{allow}\left(-0.5294\sigma_1^{allow}\right) = (250MPa)^2$$

$$1.810\left(\sigma_1^{allow}\right)^2 = (250MPa)^2$$

$$\sigma_1^{allow} = \frac{250MPa}{\sqrt{1.810}}$$

$$\sigma_1^{allow} = 185.8MPa$$

As usually, $SF = \dfrac{\text{Failure Strength}}{\text{Actual Stress}}$

$$SF = \frac{\sigma_1^{allow}}{\sigma_1}$$

$$SF = \frac{185.8MPa}{85MPa} = 2.19$$

On the other hand,

$$SF = \frac{\sigma_3^{allow}}{\sigma_3} = \frac{-0.5294\sigma_1^{allow}}{-45MPa} = \frac{-0.5294(185.8MPa)}{-45MPa} = 2.19$$

NOTE: Again, note that both Safety Factors are exactly the same. This is expected!

Problem #3:

Part a) K_I

$$K_I = \sigma_0\sqrt{\pi \cdot a} = (30ksi)\sqrt{\pi(2in)} = 75.20Ksi\sqrt{in}$$

Part b) COD

$$COD = 2u_y\Big|_{\substack{r=a\\ \theta=\pi}}$$

$$u_y = \frac{K_I}{8\mu}\sqrt{\frac{2r}{\pi}}\left[(2\kappa+1)Sin\left(\frac{\theta}{2}\right) - Sin\left(\frac{3\theta}{2}\right)\right]$$

For plane stress, $\kappa = \frac{3-\nu}{1+\nu} = \frac{3-0.3}{1+0.3} = 2.0769$. And μ is just the Shear Modulus, $\mu = \frac{E}{2(1+\nu)} = \frac{30Msi}{2(1+0.3)} = 11.538Msi$. Substituting everything back in the equation,

$$u_y = \frac{75.20x10^3}{8(11.538x10^6)}\sqrt{\frac{2(2)}{\pi}}\left[(2(2.0769)+1)Sin\left(\frac{\pi}{2}\right) - Sin\left(\frac{3\pi}{2}\right)\right]$$

$$u_y = 5.657x10^{-3}\,in$$

Finally,

$$COD = 2u_y = 2(5.656x10^{-3}\,in) = 1.131x10^{-2}\,in$$

Part c) a_c

$$K_{IC} = \sigma_0\sqrt{\pi \cdot a_c}$$

$$a_c = \left(\frac{K_{IC}}{\sigma_0}\right)^2\frac{1}{\pi}$$

$$a_c = \left(\frac{110Ksi\sqrt{in}}{30Ksi}\right)^2\frac{1}{\pi}$$

$$a_c = 4.279in$$

$$\text{max. permissible crack length} = 2\text{a}_\text{c} = 8.56in$$

Problem #4:

Part a) N$_f$

First, we can realize that $\sigma_{\min} = R\sigma_{\max} = 20Ksi$. And we need to recalculate a_c,

$$a_c = \left(\frac{K_{IC}}{\sigma_0}\right)^2 \frac{1}{\pi}$$

$$a_c = \left(\frac{110Ksi\sqrt{in}}{40Ksi}\right)^2 \frac{1}{\pi}$$

$$a_c = 2.407in$$

Then we just need to integrate the crack growth law,

$$\frac{da}{dN} = 1.6x10^{-12}(\Delta K)^4$$

$$dN = \frac{da}{1.6x10^{-12}(\Delta K)^4}$$

However, ΔK is not a constant but it's a function of the crack length $\Delta K = \Delta\sigma\sqrt{\pi \cdot a}$,

$$dN = \frac{da}{1.6x10^{-12}\left(\Delta\sigma\sqrt{\pi \cdot a}\right)^4}$$

$$dN = \frac{da}{1.6x10^{-12}\Delta\sigma^4\pi^2 a^2}$$

Now integrating,

$$\int_{N_0}^{N_f} dN = \int_{a_0}^{a_f} \frac{da}{1.6x10^{-12}\Delta\sigma^4\pi^2 a^2}$$

$$N_f - N_0 = \frac{1}{1.6x10^{-12}\Delta\sigma^4\pi^2}\left[\frac{1}{a_0} - \frac{1}{a_f}\right]$$

$$N_f - 0 = \frac{1}{1.6x10^{-12}(20Ksi)^4\pi^2}\left[\frac{1}{2in} - \frac{1}{2.407in}\right]$$

NOTE: The units of ΔK have to be Ksi not psi.

$$N_f = 33{,}480 cycles$$

Part b) Residual strength after 10,000 cycles

Now, $N_f = 10{,}000 cycles$

$$N_f - N_0 = \frac{1}{1.6x10^{-12}\Delta\sigma^4\pi^2}\left[\frac{1}{a_0} - \frac{1}{a_f}\right]$$

$$10{,}000 cycles = \frac{1}{1.6x10^{-12}(20Ksi)^4\pi^2}\left[\frac{1}{2in} - \frac{1}{a_f}\right]$$

From here, we can calculate a_f after 10,000 cycles,

$$a_f = 2.1064in$$

And its corresponding residual stress after 10,000 cycles is

$$K_{IC} = \sigma_{RES}\sqrt{\pi \cdot a}_f$$

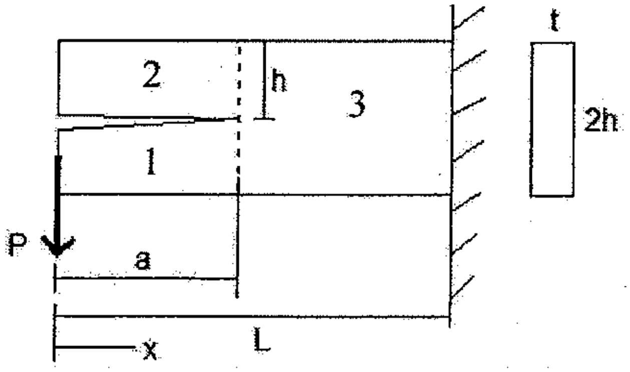
$$\sigma_{RES}^2 = \frac{K_{IC}^2}{\pi \cdot a_f}$$

$$\sigma_{RES} = \sqrt{\frac{\left(110Ksi\sqrt{in}\right)^2}{\pi(2.1064in)}}$$

$$\sigma_{RES} = 42.76Ksi$$

Problem #5:

Part a) Figure 6.25,



The strain energy stored in the beam due to bending moment M is,

$$U = \int_0^L \frac{M^2}{2EI} dx$$

For segment 1, the strain energy is,

$$U_1 = \int_0^a \frac{(Px)^2}{2EI_1} dx = \frac{P^2}{2EI_1} \frac{x^3}{3}\Bigg|_0^a = \frac{P^2 a^3}{6EI_1} = \frac{2P^2 a^3}{Eth^3} \qquad \text{where, } I_1 = \frac{th^3}{12}$$

For segment 2,

$$U_2 = 0$$

And for segment 3,

$$U_3 = \int_a^L \frac{(Px)^2}{2EI_3} dx = \frac{P^2}{2EI_3} \frac{x^3}{3}\Bigg|_a^L = \frac{P^2\left(L^3 - a^3\right)}{6EI_3} = \frac{P^2\left(L^3 - a^3\right)}{4Eth^3} \text{ where, } \qquad I_3 = \frac{t(2h)^3}{12} = \frac{2th^3}{3}$$

Or we could change the coordinate system defining "x" from the tip of the crack. In that way, "x" varies from 0 to L-a. However, in this case, we need to consider a moment of magnitude P.a generated because the translocation of the load.

$$U_3 = \int_0^{L-a} \frac{(P \cdot a + P \cdot x)^2}{2EI_3} dx = \int_0^{L-a} \frac{P^2}{2EI_3}\left(a^2 + 2ax + x^2\right)dx = \frac{P^2}{2EI_3}\left(a^2 x + ax^2 + \frac{x^3}{3}\right)_0^{L-a}$$

$$\frac{P^2}{2EI_3}\left(a^2(L-a)+a(L-a)^2+\frac{(L-a)^3}{3}\right)$$

$$\frac{P^2}{2EI_3}\left(a^2L-a^3+aL^2-2a^2L+a^3+\frac{L^3}{3}-L^2a+La^2-\frac{a^3}{3}\right)$$

$$\frac{P^2}{2EI_3}\left(\frac{L^3}{3}-\frac{a^3}{3}\right)$$

or

$$\frac{P^2}{6EI_3}\left(L^3-a^3\right) \quad \text{Just as before.!!!!}$$

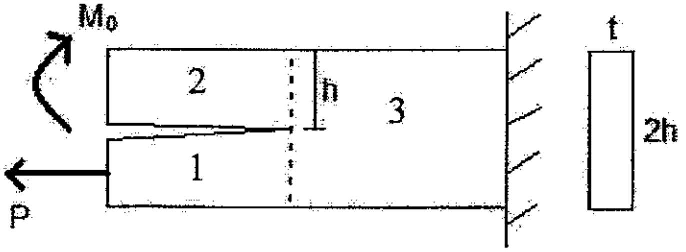
Finally, the total strain energy stored in the entire split beam is

$$U=U_1+U_2+U_3=\frac{2P^2a^3}{Eth^3}+0+\frac{P^2\left(L^3-a^3\right)}{4Eth^3}$$

The strain energy rate is

$$G=\frac{1}{t}\frac{dU}{da}=\frac{1}{t}\left(\frac{6P^2a^2}{Eth^3}-\frac{3P^2a^2}{4Eth^3}\right)=\frac{21}{4}\frac{P^2a^2}{Et^2h^3}$$

Part b) Figure 6.26,



The strain energy stored in the beam due to an Axial load P is

$$U=\frac{P^2L}{2EA}$$

For segment 1, the strain energy is

$$U_1 = \frac{P^2 a}{2EA_1} = \frac{P^2 a}{2Eth} \qquad \text{where, } A_1 = th$$

For segment 2,

$$U_2 = \int_0^a \frac{M_0^2}{2EI_2}dx = \frac{M_0^2 a}{2EI_2} = \frac{6M_0^2 a}{Eth^3} \qquad \text{where, } I_2 = \frac{th^3}{12}$$

For segment 3,

$$U_3 = \frac{P^2(L-a)}{2EA_3} + \int_a^L \frac{\left(M_0 + P\frac{h}{2}\right)^2}{2EI_3}dx \qquad \text{where, } \begin{cases} I_3 = \frac{t\cdot(2h)^3}{12} = \frac{2th^3}{3} \\ A_3 = 2th \end{cases}$$

$$U_3 = \frac{P^2(L-a)}{4Eth} + \int_a^L \frac{3\left(M_0^2 + M_0 Ph + \frac{P^2h^2}{4}\right)}{4Eth^3}dx$$

$$U_3 = \frac{P^2(L-a)}{4Eth} + \frac{3M_0^2(L-a)}{4Eth^3} + \frac{3M_0 Ph(L-a)}{4Eth^3} + \frac{3P^2h^2(L-a)}{16Eth^3}$$

The total strain energy stored in the entire split beam is

$$U = U_1 + U_2 + U_3$$

$$U = \frac{P^2 a}{2Eth} + \frac{6M_0^2 a}{Eth^3} + \frac{P^2(L-a)}{4Eth} + \frac{3M_0^2(L-a)}{4Eth^3} + \frac{3M_0 Ph(L-a)}{4Eth^3} + \frac{3P^2h^2(L-a)}{16Eth^3}$$

The strain energy release rate is

$$G = \frac{1}{t}\frac{dU}{da} = \frac{1}{t}\left[\frac{P^2}{2Eth} + \frac{6M_0^2}{Eth^3} - \frac{P^2}{4Eth} - \frac{3M_0^2}{4Eth^3} - \frac{3M_0 Ph}{4Eth^3} - \frac{3P^2h^2}{16Eth^3}\right]$$

$$G = \frac{1}{4}\frac{P^2}{Et^2h} + \frac{21}{4}\frac{M_0^2}{Et^2h^3} - \frac{3M_0 Ph}{4Et^2h^3} - \frac{3P^2h^2}{16Et^2h^3}$$