Transient Heat Conduction

Temperature in objects in general can vary with position and time
$T(x,y,z,t)$

Lumped System Analysis:

Temperature varies with time, but is uniform throughout the body

Examples:
- A hot copper ball removed from an oven
- An egg placed in boiling water
- A potato being baked in an oven

(May not satisfy analysis — the egg and potato examples)

From the Conservation of Energy:

Heat transfer into a body during Δt = Increase in energy of the body during Δt

$$h A_s (T_\infty - T)\, dt = m C_p\, dT \qquad \text{where } m = \rho V$$

$$\text{Let} \quad \theta = T - T_\infty \,, \quad \theta_i = T_i - T_\infty \quad \text{and} \quad b = \frac{h A_s}{\rho V C_p}$$

$$L_c = \text{Characteristic length} = V/A_s$$

$$d\theta = -b\,\theta\, dt \qquad \text{or} \qquad \int_{\theta_i}^{\theta} \frac{d\theta}{\theta} = \int_0^t -b\, dt$$

$$\frac{T - T_\infty}{T_i - T_\infty} = e^{-bt}$$

The accuracy of the lumped system assumption can be obtained through the Biot Number (Bi), $Bi = h/(k/L_c)$

$$Bi = \frac{L_c/k}{1/h} = \frac{\text{Conduction Resistance Within}}{\text{Convection Resistance Outside}} \qquad Bi \le 0.10$$

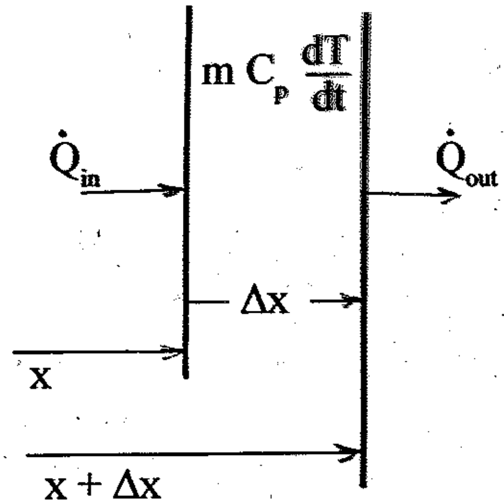
Temperature Variations with time and position in one-dimensional problems

Plane wall of thickness 2L:

$$\dot{Q}_{in} = -kA \left.\frac{dT}{dx}\right)_x$$

$$\dot{Q}_{out} = -kA \left.\frac{dT}{dx}\right)_{x+\Delta x}$$

For Plane Wall



Differential equation: $\dfrac{\partial^2 T}{\partial x^2} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial t}$

Boundary conditions: $\dfrac{\partial T(0, t)}{\partial x} = 0$ and

$$-k\frac{\partial T(L, t)}{\partial x} = h[T(L, t) - T_\infty]$$

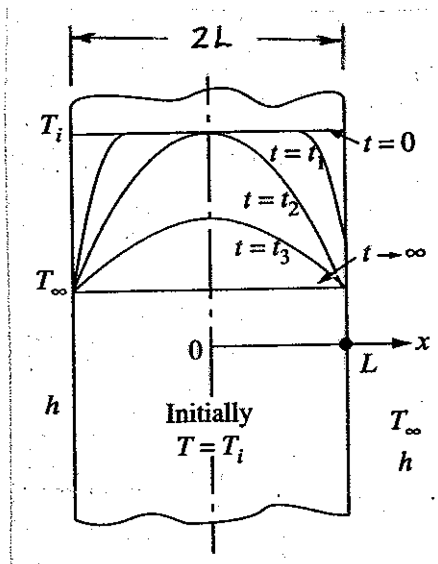
Initial condition: $T(x, 0) = T_i$

$$\theta_{wall} = \frac{T(x, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \cos(\lambda_1 x/L), \quad \tau > 0.2$$

Center of plane wall ($x = 0$):

$$\theta_{0,\,wall} = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$

$$\left(\frac{Q}{Q_{max}}\right)_{wall} = 1 - \theta_{0,\,wall}\frac{\sin \lambda_1}{\lambda_1}$$



Tabular and Graphical Solutions

Partial differential equations are also a results of formulating temperature variations with r and t within cylinders and spheres, the solutions of which normally involve mathematics beyond the undergraduate.

As a consequence, tabular and graphical solutions are presented as well as approximations, but require a series of dimensionless quantities

$$\theta(x,t) = \frac{T(x,t) - T_\infty}{T_i - T_\infty} \qquad X = x/L$$

$$Bi = hL/k \qquad \tau = \alpha\, t/L^2$$

For $\tau > 0.2$, good approximations can be obtained for the plane wall, cylinder and sphere using the relationships below. Replace L with r_o in determining the dimensionless numbers for cylinders and spheres.

Plane wall: $$\theta(x,t)_{\text{wall}} = \frac{T(x,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \cos(\lambda_1 x/L),$$

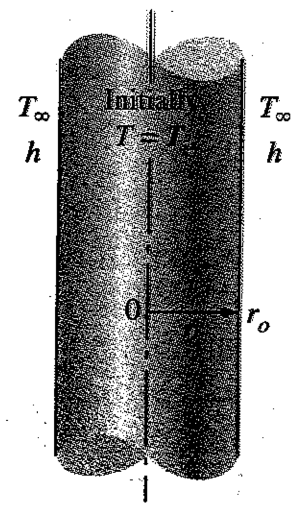
Cylinder: $$\theta(x,t)_{\text{cyl}} = \frac{T(r,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} J_0(\lambda_1 r/r_0),$$

Sphere: $$\theta(x,t)_{\text{sph}} = \frac{T(r,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \frac{\sin(\lambda_1 r/r_0)}{\lambda_1 r/r_0},$$

Temperature Variations with time and position in one-dimensional problems

Cylinder:

$$\theta_{cyl} = \frac{T(r, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} J_0(\lambda_1 r/r_o), \quad \tau > 0.2$$

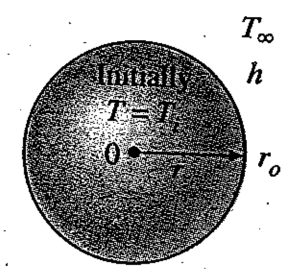


Center of cylinder ($r = 0$): $$\theta_{0,\,cyl} = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$

$$\left(\frac{Q}{Q_{max}}\right)_{cyl} = 1 - 2\theta_{0,\,cyl} \frac{J_1(\lambda_1)}{\lambda_1}$$

Sphere:

$$\theta_{sph} = \frac{T(r, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \frac{\sin(\lambda_1 r/r_o)}{\lambda_1 r/r_o}, \quad \tau > 0.2$$



Center of sphere ($r = 0$):

$$\theta_{0,\,sph} = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$

$$\left(\frac{Q}{Q_{max}}\right)_{sph} = 1 - 3\theta_{0,\,sph} \frac{\sin \lambda_1 - \lambda_1 \cos \lambda_1}{\lambda_1^3}$$

Transient Heat Conduction in Semi-infinite Solids

Case 1: Surface Temperature T_s = constant

$$\frac{T(x,t) - T_i}{T_s - T_i} = \text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \quad \text{and} \quad \dot{q}_s(t) = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}}$$

Case 2: Heat Flux $\dot{q}_s$ = constant

$$T(x,t) - T_i = \frac{\dot{q}_s}{k}\left[\sqrt{\frac{4\alpha t}{\pi}}\exp\left(-\frac{x^2}{4\alpha t}\right) - x\,\text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)\right]$$

Case 3: Convection at the surface $\dot{q}_s = h\,[(T_\infty - T(0,t)]$.

$$\frac{T(x,t) - T_i}{T_\infty - T_i} = \text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - \exp\left(\frac{hx}{k} + \frac{h^2\alpha t}{k^2}\right)\text{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)$$