Math 4322

SOLUTIONS TO PRACICE EXAM I.

1. (a) 5-i.
(b) 14-5i.

2. $\overline{\dfrac{(1+2i)^2}{3-4i}} = \dfrac{-3-4i}{3-4i} = \dfrac{(-3-4i)(3+4i)}{25} = \dfrac{-7-24i}{25}.$

3. $\dfrac{i}{-2-2i} = \dfrac{-1-i}{4}$. Answer: $\dfrac{5\pi}{4}$.

4. $(-1+i)^{100} = \left(\sqrt{2}\exp\left[i\left(\dfrac{3\pi}{4}\right)\right]\right)^{100} = 2^{50}\exp[i(75\pi)] = 2^{50}\exp[i\pi] = -2^{50}.$

5. $\left(-\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i\right)^{\frac{1}{2}} = \exp[i(2\pi/3)]$. Then $(\exp[i(2\pi/3)])^{1/2} = \exp[i(\pi/3+\pi n)] =$ $\pm\exp[i(\pi/3)] = \pm\left(\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i\right)$.

6. $\text{Re}\dfrac{x+yi-(1+i)}{x+yi} = \text{Re}\dfrac{((x-1)+(y+1)i)(x-yi)}{x^2+y^2} = \dfrac{x^2+y^2-x+y}{x^2+y^2} > 0.$ I.e. need to sketch the set of points (x,y) with $x^2+y^2-x+y>0$. Same as $(x-\frac{1}{2})^2+(y+\frac{1}{2})^2>\frac{1}{2}$. These are points outside of the circle of radius 1/2 and center $(1/2,-1/2)$.

7. Writing $f(z)=u(x,y)+iv(x,y)$, get $u(x,y)=x, v(x,y)=0$. CR equations fail at every point as $u_x=1$ and $v_y=0$.

8. $u(x,y)=e^{-y}\cos x, v(x,y)=e^{-y}\sin x$. The partial derivatives are: $u_x = -e^{-y}\sin x, u_y=-e^{-y}\cos x, v_x=e^{-y}\cos x, v_y=-e^{-y}\sin x$. They are continuous everywhere and satisfy CR equations. Thus, $f(z)$ is differentiable everywhere (= is entire). $f'(z)=-e^{-y}\sin x+ie^{-y}\cos x=if(z)$.

9. $e^ye^{ix}=e^y\cos x+ie^y\sin x$. $u(x,y)$ and $v(x,y)$ do not satisfy CR equations: e.g. $u_x=-e^y\sin x$ and $v_y=e^y\sin x$.

10. (a) $u_{xx}=6x$, $u_{yy}=-6x$, so $u_{xx}+u_{yy}=0$.
(b) Let $f(z)=u(x,y)+iv(x,y)$. $f(z)$ is entire, so $v_y=u_x=3x^2-3y^2$, $v_x=-u_y=6xy$. From the first equality, $v=\int 3x^2-3y^2\,dy=3x^2y-y^3+g(x)$. Comparing with the second equation, we see that $g'(x)=0$, thus $g(x)$ is a constant. Answer $f(z)=u(x,y)+iv(x,y)$, where $v(x,y)=3x^2y-y^3+c$.

11. $-2i=2e^{(3\pi/2)i}$. $\log(-2i)=\ln 2+i(3\pi/2+2n\pi)$.

12. $\exp(x+i\pi/4)=e^x\exp(i\pi/4)=e^x(cos\frac{\pi}{4}+i\sin\frac{\pi}{4})=\frac{\sqrt{2}}{2}e^x(1+i)$. $\frac{\sqrt{2}}{2}e^x$ can assume any positive value, thus the image on the (u,v) plane is a ray (u,v), $u>0$.

13. $z=\sin^{-1}2=-i\log\left(2i+(1-4)^{1/2}\right)$. $(-3)^{1/2}=\pm i\sqrt{3}$, so we have to compute $-i\log(i(1\pm\sqrt{3}))$. $-i\log(i(1+\sqrt{3}))=-i\ln(1+\sqrt{3})+(\pi/2+2\pi n)$ and

$-i\log(i(1-\sqrt{3})) = -i\ln(\sqrt{3}-1)+(3\pi/2+2\pi n) = i\ln(\sqrt{3}-1)+(\pi/2+(2n+1)\pi)$. Thus $\sin^{-1} 2 = \pi/2 + \pi m - i\ln(\sqrt{3}+(-1)^m)$.

14. We have to exclude z such that $z^2+1=0$ (otherwise the denominator is 0) and $(z-1)$ is real and nonnegative (otherwise Log is not defined). Thus the function is defined in the domain $\{z = x+iy \,|\, z \neq \pm i; z \neq x \text{ whenever } x \geq 1\}$. Whenever defined, the function is analytic because it is a ration of two differentiable functions and the denominator is nonzero.

15. $i^i = \exp(i\log i)$, hence the principle value is $\exp(i\,\mathrm{Log}\,i)$. $\mathrm{Log}\,i = \ln 1 + i\pi/2$, thus $\exp(i\,\mathrm{Log}\,i) = \exp(i(i\pi/2)) = \exp(-\pi/2)$.