Math 4322

SOLUTIONS TO PRACICE EXAM II.

1. Parametrization of C: $z = t + it$, $0 \le t \le 1$. On C, $\bar{z} = t - it$. $\int_C \bar{z}dz = \int_0^1 (t - it)(t + it)'dt = \int_0^1 (t - it)(1 + i)dt = \int_0^1 2tdt = 1$.

2. z^3 is analytic, can use its antiderivative: $\int_C z^3 dz = \left.\frac{z^4}{4}\right|_1^3 = \frac{81}{4} - \frac{1}{4} = 20$.

3. a) $\int_C \frac{\cos z}{z} dz = 2\pi i \cos 0 = 2\pi i$ (Cauchy integral formula).

b) Same answer as in (a): can change contours because $\frac{\cos z}{z}$ is analytic between them.

4. $\int_C \frac{e^z}{(z-\pi i)^2} dz = \frac{2\pi i}{1!} \frac{de^z}{dz}(\pi i) = 2\pi i e^{\pi i} = -2\pi i$ (we use $f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz$).

5. By maximum modulus principle, if $f(z)$ is non-constant, it attains the maximum only on the boundary. Since this is not the case here ($|f(0)|$ is maximal), $f(z)$ must be constant. So, $f(z) = -3$.

6. $\frac{1}{3 - z^2} = \frac{1}{3} \frac{1}{1 - (z^2/3)} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{z^2}{3}\right)^n = \sum_{n=0}^{\infty} \frac{1}{3^{n+1}} z^{2n}$. Radius of convergence: $\left|\frac{z^2}{3}\right| < 1$, so $|z| < \sqrt{3}$.

7. $\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1} = -\frac{1}{1-(z/2)} - \frac{1}{z}\frac{1}{1-(1/z)}$

$= -\sum_{n=0}^{\infty} \frac{z^n}{2^n} - \frac{1}{z}\sum_{k=0}^{\infty} \frac{1}{z^k} = \sum_{n=0}^{\infty} -\frac{1}{2^n} z^n + \sum_{m=1}^{\infty} -z^{-m}$.

8. $\frac{\sin z}{(z^2-1)(z-2)} = \frac{\sin z}{(z-1)(z+1)(z-2)}$,

$\text{Res}_{z=1} \frac{\sin z}{(z-1)(z+1)(z-2)} = \frac{\sin 1}{(1+1)(1-2)} = -\frac{\sin 1}{2}$.

9. $\int_C \frac{3z^2+2}{(z-1)(z-2)^2} dz = 2\pi i \left(\text{Res}_{z=1} \frac{3z^2+2}{(z-1)(z-2)^2} + \text{Res}_{z=2} \frac{3z^2+2}{(z-1)(z-2)^2} \right) = 6\pi i$.

$\text{Res}_{z=1} \frac{3z^2+2}{(z-1)(z-2)^2} = \left.\frac{3z^2+2}{(z-2)^2}\right|_1 = \frac{3(1)^2+2}{(1-2)^2} = 5$ (simple pole).

$\text{Res}_{z=2} \frac{3z^2+2}{(z-1)(z-2)^2} = \left.\frac{d}{dz}\left(\frac{3z^2+2}{z-1}\right)\right|_2 = -\left.\frac{2}{(z-1)^2}\right|_2 = -\frac{2}{(2-1)^2} = -2$ (pole of order 2).