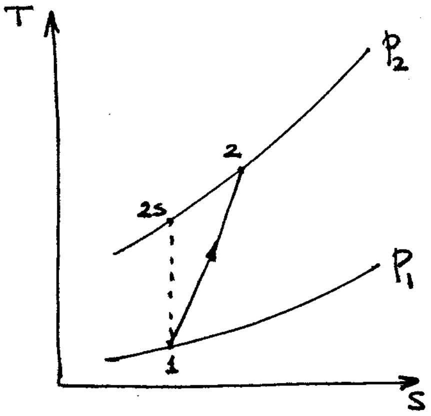
Need. Show:

(1) $w_s = w_{min}$ for adiabatic steady flow compression

(2) $\eta_c \equiv \dfrac{w_s}{w_{actual}}$ is reasonable

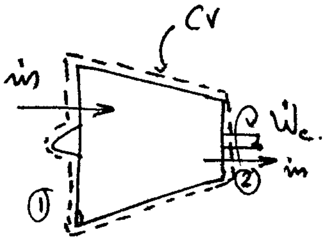
(3) Reasonable definition of η_t for a turbine.



Given: Adiabatic, steady flow compression

Initial state (p_1, T_1)

Solution



Need "specific work" $w = \dfrac{\dot W}{\dot m}$ for adiabatic process

Since we are dealing with energy transfers, use 1st law for the CV.

$$\cancel{\dot Q} + \dot W_c \overset{(1)}{=} \dot m\,(h_{02} - h_{01})$$

0 (given)

$$\overset{(2)}{=} \dot m\,(h_2 - h_1)$$

$$\overset{(3)}{=} \dot m\, c_p\,(T_2 - T_1)$$

Assumptions.

(1) Quasi 1-D flow

(2) No info on KE or velocity, so neglect $h_0 \simeq h$

(3) Ideal gas, const c_p

$$w = \frac{\dot W_c}{\dot m} = c_p\,(T_2 - T_1) = c_p T_1 \left(\frac{T_2}{T_1} - 1\right) = w_{actual}$$

To get an idea of limiting case, we recognize that a limiting case is the reversible compression process. Use Gibbs' equation to connect entropy changes to changes in other variables.

$$s_2 - s_1 \equiv \Delta s = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1} \geq 0$$ for an adiabatic process from 2nd Law of Thermodynamics

Rearranging we have

$$\ln \frac{T_2}{T_1} = \frac{\Delta s}{c_p} + \frac{R}{c_p} \ln\left(\frac{p_2}{p_1}\right) \qquad \frac{R}{c_p} = \frac{\gamma - 1}{\gamma}$$

$$\ln \frac{T_2}{T_1} = \frac{\Delta s}{c_p} + \ln\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}$$

or

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} \cdot \exp\left(\frac{\Delta s}{c_p}\right) \quad ; \ \Delta s \geq 0$$

Now for $x \geq 0$ the minimum value of $e^x = 1$ when $x = 0$

So $\frac{T_2}{T_1}$ is minimum when $\Delta s = 0$ (for given $\frac{P_2}{P_1}$).

Thus $w_{actual} = c_p T_1 \left(\frac{T_2}{T_1} - 1 \right)$ is a minimum when $\Delta s = 0$ ← (1)

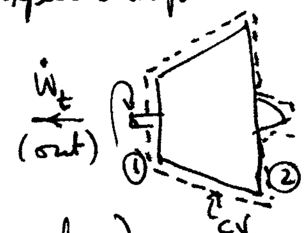
and $w_s = c_p T_1 \left[\left(\frac{P_2}{P_1} \right)^{\frac{\gamma-1}{\gamma}} - 1 \right]$

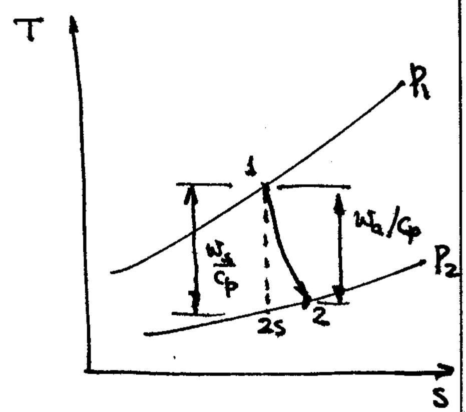
$$w_{actual} = c_p T_1 \left[\left(\frac{P_2}{P_1} \right)^{\frac{\gamma-1}{\gamma}} \exp\left(\frac{\Delta s}{c_p} \right) - 1 \right] \geq w_s .$$

Hence a reasonable efficiency definition that makes $\eta \leq 1$ is

$$\eta_c = \frac{w_s}{w_{actual}} = \frac{(P_2/P_1)^{\frac{\gamma-1}{\gamma}} - 1}{\frac{T_2}{T_1} - 1} \qquad \leftarrow (2)$$

In a turbine we have an expansion process





1st Law $\cancel{\dot{Q}} + \underset{(<0)}{\dot{W}_t} \overset{(1)}{=} \dot{m}(h_{02} - h_{01})$

0 (given)

$\overset{(2)}{=} \dot{m}(h_2 - h_1)$

$\overset{(3)}{=} \dot{m} c_p T_1 \left(\frac{T_2}{T_1} - 1 \right)$ $\quad T_2 < T_1$ so $\dot{W}_t < 0$.

$$w_t = \frac{\dot{W}_t}{\dot{m}} = c_p T_1 \left(\frac{T_2}{T_1} - 1 \right)$$

Gibbs' eq'n $\Delta s \equiv s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{P_2}{P_1}$ as before

and once again we have

$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1} \right)^{\frac{\gamma-1}{\gamma}} \exp\left(\frac{\Delta s}{c_p} \right) ; \quad \Delta s \geq 0$$

Once again $\Delta s = 0$ gives the minimum value of $T_2 = T_{2s}$

But $w_t = c_p T_1 \left(\frac{T_2}{T_1} - 1 \right) = -c_p T_1 \left(1 - \frac{T_2}{T_1} \right) < 0$ output

$|w_t| = c_p T_1 \left(1 - \frac{T_2}{T_1} \right)$ so minimum value of $T_2 = T_{2s}$ gives maximum of $|w_t|$

Thus the largest magnitude of specific work output corresponds to the isentropic expansion in a turbine.

$$|w_{ts}| = c_p T_1 \left(1 - \frac{T_{2s}}{T_1}\right) = c_p T_1 \left[1 - \left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}}\right]$$

$$\left\{ \text{Note} \quad w_{ts} = \ominus c_p T_1 \left[1 - \left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}}\right] \right\}.$$

and a reasonable definition of adiabatic efficiency of expansion for a turbine $\eta_t = \dfrac{|w_t|}{|w_{ts}|}$ because $|w_t| \leq |w_{ts}|$

$$\eta_t \equiv \frac{1 - (P_2/P_1)^{\frac{\gamma-1}{\gamma}}}{1 - \frac{T_2}{T_1}} \qquad \longleftarrow (3)$$

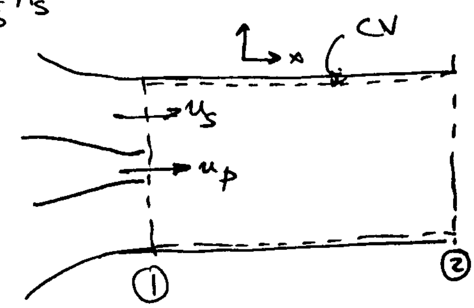
Need. (a) Flow properties at ②

(b) T_2, P_2, u_2

Given: Constant area mixing tube shown on sketch.

Air. ${}_p = 300\,m/s$, $T_p = 900K$, $u_s = 30\,m/s$, $T_s = 300K$.

$p_1 = 0.1\,MPa$ $\quad A_p = \frac{1}{3} A_s$



Solution.
For simplicity we assume quasi 1-D flow at inlet and exit of CV shown (given)
To calculate flow properties we use mass, momentum energy equations and equation of state.

Assumptions.
(1) Steady flow.
(2) Ideal gas (given)
(3) Frictionless.
(4) Adiabatic

Mass. $\quad \dot{m}_2 \underset{(1)}{=} \dot{m}_1 = \dot{m}_p + \dot{m}_s$

$$\rho_2 u_2 A_2 = \rho_p u_p A_p + \rho_s u_s A_s$$

From geometry of constant area mixer

$$A_2 = A_p + A_s = A_p + 3A_p = 4A_p .$$

so $\quad 4\rho_2 u_2 = \rho_p u_p + 3\rho_s u_s . \Rightarrow \rho_2 u_2 = \frac{1}{4}\rho_p u_p + \frac{3}{4}\rho_s u_s \rightarrow ①$

x-Momentum:

$$\underset{0(3)}{\cancel{F_f}} + p_1(A_p + A_s) - p_2 A_2 = \dot{m}_2 u_2 - \dot{m}_p u_p - \dot{m}_s u_s .$$

$$4A_p(p_1 - p_2) = \rho_2 u_2^2\, 4A_p - \rho_p u_p^2 A_p - 3\rho_s u_s^2 A_p .$$

$$\rho_2 u_2^2 + p_2 = p_1 + \frac{1}{4}\rho_p u_p^2 + \frac{3}{4}\rho_s u_s^2 \quad \rightarrow ②$$

Energy

$$\underset{0(4)}{\cancel{\dot{Q}}} + \underset{0(\text{mixer - no power transfer})}{\cancel{\dot{W}_s}} = \dot{m}_2\left(h_2 + \frac{u_2^2}{2}\right) - \dot{m}_p\left(h_p + \frac{u_p^2}{2}\right) - \dot{m}_s\left(h_s + \frac{u_s^2}{2}\right)$$

$$h_2 + \frac{u_2^2}{2} = \frac{\dot{m}_p(h_p + u_p^2/2) + \dot{m}_s(h_s + \frac{u_s^2}{2})}{\dot{m}_2}$$

$$h_2 + \frac{u_2^2}{2} = \frac{\rho_p u_p (h_p + u_p^2/2) + 3\rho_s u_s (h_s + u_s^2/2)}{(\rho_p u_p + 3\rho_s u_s)}$$

using mass conservation for $\dot{m}_2$

$$c_p T_2 + \frac{u_2^2}{2} = \frac{\rho_p u_p (c_p T_p + \frac{u_p^2}{2}) + 3\rho_s u_s (c_p T_s + \frac{u_s^2}{2})}{(\rho_p u_p + 3\rho_s u_s)} \longrightarrow (3)$$

Assumptions (cont'd)

(5) Constant specific heat, so $h = c_p T$

(6) $R_{air} = 287\ J/kg\,K$; $\gamma = 1.4$

In these 3 equations, all conditions on the right hand side are specified except ρ_p and ρ_s which can be calculated using the ideal gas eq'n of state : $\rho = \frac{p}{RT}$

On the left hand side we have unknowns p_2, T_2, u_2 (ρ_2 can be calculated from $\rho_2 = \frac{p_2}{RT_2}$). Hence we have 3 equations in 3 unknowns and we can determine conditions at (2) from known conditions at (1). In general need to solve numerically. From 1, 2, 3 we have

$$\rho_2 u_2 = \frac{1}{4}\rho_p u_p + \frac{3}{4}\rho_s u_s \equiv \alpha_1 = 55.2\ \frac{kg}{m^2 s}$$

$$p_2 + \rho_2 u_2^2 = p_1 + \frac{1}{4}\rho_p u_p^2 + \frac{3}{4}\rho_s u_s^2 \equiv \alpha_2 = 1.095 \times 10^5\ N/m^2$$

$$T_2 + \frac{u_2^2}{2c_p} = \frac{\rho_p u_p (T_p + \frac{u_p^2}{2c_p}) + 3\rho_s u_s (T_s + \frac{u_s^2}{2c_p})}{(\rho_p u_p + 3\rho_s u_s)} \equiv \alpha_3 = 640\,K$$

Could solve iteratively, but can set this up as a quadratic equation in u_2.

(1) $\Rightarrow \rho_2 u_2 = \alpha_1 \Rightarrow \frac{p_2}{RT_2} u_2 = \alpha_1$ or $T_2 = \frac{p_2}{\alpha_1 R} u_2$.

(2) $\Rightarrow p_2 = \alpha_2 - \alpha_1 u_2$

(3) $\Rightarrow \frac{p_2 u_2}{\alpha_1 R} + \frac{u_2^2}{2c_p} = \alpha_3 \Rightarrow \frac{u_2^2}{2c_p} + \frac{(\alpha_2 - \alpha_1 u_2) u_2}{\alpha_1 R} - \alpha_3 = 0$

$$u_2^2 \left(1 - \frac{R}{2c_p}\right) - u_2 \frac{\alpha_2}{\alpha_1} + R\alpha_3 = 0.$$

using $\frac{R}{c_p} = \frac{\gamma - 1}{\gamma}$ $\quad 1 - \frac{R}{2c_p} = \frac{\gamma+1}{2\gamma}$

So finally

$$\frac{\gamma+1}{2\gamma} u_2^2 - \frac{\alpha_2}{\alpha_1} u_2 + R\alpha_3 = 0$$

u_2 can be obtained as a solution of this quadratic equation.

For the numerical values specified

$\frac{\gamma+1}{2\gamma} = 0.857.$ $\quad -\frac{\alpha_2}{\alpha_1} = -1985 \ \frac{m}{s}$ $\quad R\alpha_3 = 640 \ \frac{m^2}{s^2}$

Solving quadratic we get $u_2 = 2.22 \frac{km}{s}$ or $96.5 \frac{m}{s}$

The large value of u_2 gives negative (and hence obviously unphysical) values of T_2 { from $T_2 = \alpha_3 - \frac{u_2^2}{2c_p}$). Hence this must be discarded.

Finally $u_2 = 96.5$ m/s.

$T_2 = 635$ K

$p_2 = 104$ kPa

$\rho_2 = 0.572$ kg/m³

Need: (a) T-s diagram with constant pressure lines

(b) Nozzle exit velocity.

Given: Data on problem sheet for s_i, T_i, p_i — i - various states in an afterburning turbojet.

$$\frac{T}{T_{ref}} = \left(\frac{p}{p_{ref}}\right)^{\frac{\gamma-1}{\gamma}} \exp\left(\frac{s-s_{ref}}{c_p}\right)$$

$s_{ref} = 0$ $\quad T_{ref} = 300\,K$ $\quad p_{ref} = 100\,kPa$.

$\gamma = 1.4$ $\quad R = 287\ J/kg\,K$.

Solution.

The equation of a constant pressure line through state 03 (for example) is given by $T = T_{ref}\left(\frac{p_{03}}{p_{ref}}\right)^{\frac{\gamma-1}{\gamma}} \exp\left(\frac{s-s_{ref}}{c_p}\right)$.

$$c_p = \frac{\gamma}{\gamma-1}R = 1005\ J/kg\,K.$$

State 03 has $p_{03} = 2100\,kPa$ $\quad p_{ref} = 100\,kPa$ $\quad s_{ref} = 0\ J/kgK$.

$\Rightarrow \quad T = 300\times(21)^{3.5}\exp\left(\frac{s}{1005}\right)$ is the eqn of the line $T = T(s)$ @ $p = const = 2100\,kPa$

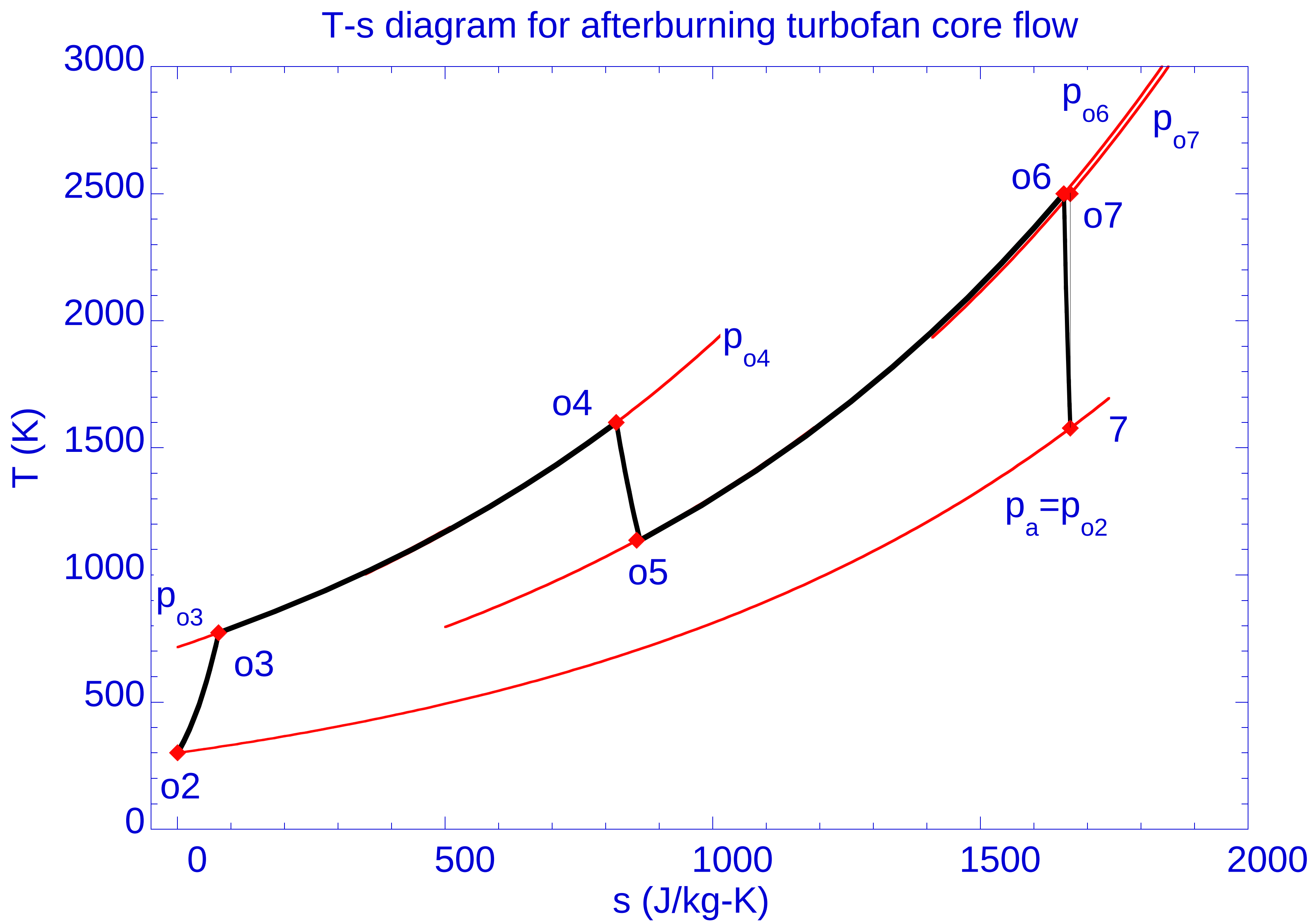
Similarly for all other states.

Nozzle outlet velocity u_7 $\qquad T_{07} = T_7 + \frac{u_7^2}{2c_p}$

$$u_7 = \sqrt{2c_p(T_{07}-T_7)}$$
$$= \sqrt{2\times 1005\frac{J}{kg\,K}(2500-1578.2)K}$$
$$= 1.36\times10^3\ m/s = 1.36\ km/s$$

$\longleftarrow$ u_7 Nozzle exit velocity

Note: Because the stagnation pressure only changes (drops) slightly between 03 and 04 or between 05, 06, 07 the corresponding constant pressure lines nearly overlap and are hard to distinguish in the graph. The difference is usually exaggerated in our T-s sketches (for clarity).



T-s diagram for afterburning turbofan core flow
3000
2500
2000
1500
1000
500
0
T (K)
0
500
1000
1500
2000
s (J/kg-K)
p o6
p o7
o6
o7
p o4
o4
7
p a =p o2
o5
p o3
o3
o2