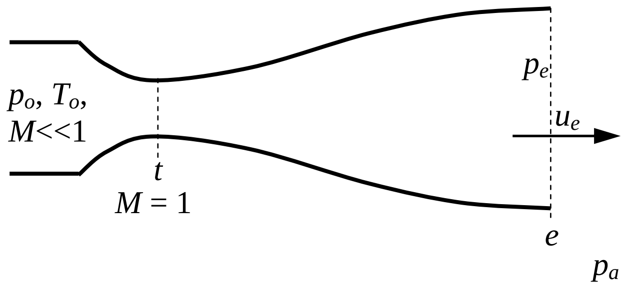
Need: (a) $\dot{m}$ (b) p_e, $\Im_{SL}$ for a rocket nozzle

Given: $\Im$ = 3.82 *MN*, $\hat{M}_{prop}$ = 15 *kg/kmol*,
γ = 1.18, $I_{sp,vac}$ = 480 *s*, H_2-O_2 propellant
Ideal nozzle, p_o = 19.8 *MPa*, T_o = 3400 *K*,
A_e/A_t = 72



Solution:

$$\Im_{vac} \overset{(1)}{=} \dot{m}u_e + \left(p_e - p_a\right)A_e = \dot{m}u_e + p_e A_e \;\; (p_a = 0 \text{ in vacuum})$$

$$I_{sp,vac} = \frac{\Im_{vac}}{\dot{m}g_o} \Rightarrow \dot{m} = \frac{\Im_{vac}}{I_{sp,vac}g_o}; \;\; g_o = 9.81 \; m/s^2 \;\; \boxed{\dot{m} = 811 \; kg/s}$$

$$\dot{m} \overset{(1)}{=} \rho_t A_t u_t; \;\; u_t = \cancel{M_{t}}_{=1}\sqrt{\gamma R_{prop} T_t}$$

$$R_{prop} \overset{(4)}{=} \frac{\hat{R}}{\hat{M}_{prop}} = \frac{8314 J/kmol-K}{15 kg/kmol} = 554 \; J/kg\text{-}K$$

Assumptions:
(1) Steady, quasi-1D flow
(2) Ideal gas behavior, with constant c_p
(3) Adiabatic nozzle
(4) Ideal nozzle (given) so flow is isentropic
(5) p_{SL} = 101 *kPa*

$$T_t \overset{(3)}{=} \frac{T_o}{\left(1 + \frac{\gamma - 1}{2}\cancel{M_t^2}_{1}\right)} = \frac{2}{\gamma + 1}T_o = 3119 \; K \Rightarrow u_t = \sqrt{\gamma R_{prop} T_t} = 1.43 \; km/s$$

$$p_t \overset{(5)}{=} \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} p_o = 11.3 \; MPa; \;\; \rho_t \overset{(4)}{=} \frac{p_t}{R_{prop}T_t} = 6.51 \; kg/m^3 \quad \text{Finally, } \boxed{A_t = \frac{\dot{m}}{\rho_t u_t} = 8.73\times10^{-2} \; m^2}$$

In practical rocket nozzles we know that exit flow is supersonic and M_t = 1. Hence $A_t = A^*$ and $A_e/A_t = 72 \Rightarrow A_e/A^* = 72$. But in an isentropic flow area ratio is a unique function of Mach number so we can compute exit Mach number from specified area ratio (by iteration using Eq. 3-15 in text or by using the isentropic flow spreadsheet posted on ERes). This gives M_e = 4.51.

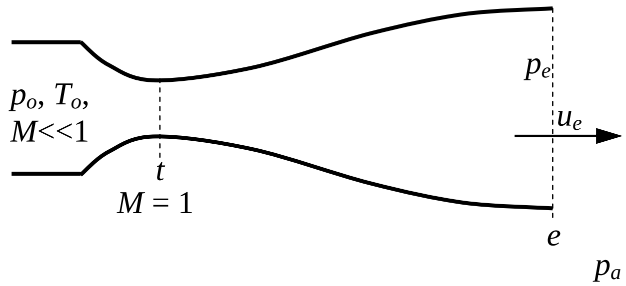
$$T_{oe} \overset{(3)}{=} T_{oi} = T_o \Rightarrow T_e = \frac{T_o}{1 + \frac{\gamma-1}{2}M_e^2} = 1200 \; K; \;\; u_e = M_e\sqrt{\gamma R_{prop} T_e} = 4.00 \; km/s$$

From the isentropic relations: $p_e = p_o\left(\frac{T_e}{T_o}\right)^{\frac{\gamma}{\gamma-1}} = \boxed{21.5 \; kPa}$

$A_e = 72 \, A_t = 6.28 \; m^2 \Rightarrow \Im_{SL} \overset{(1)}{=} \dot{m}u_e + \left(p_e - p_{a,SL}\right)A_e \overset{(5)}{=} \boxed{3.32 \; MN}$ $\Im_{SL} < \Im_{vac}$ because nozzle is overexpanded at sea-level ($p_e < p_{a,SL}$)

Need: (a) $\dot{m}$ (b) A_t (c) A_e for a rocket nozzle

Given: $\hat{M}_{prop}$ = 21.9 *kg/kmol*, γ = 1.23,
p_o = 2.15 *MPa*, T_o = 2860 *K*, p_a = 100 *kPa*,
$\Im$ = 1500 *N*, η_n = 0.96



Solution:

$$\Im \overset{(1)}{=} \dot{m}u_e + \cancel{(p_e - p_a)}_{0(2)} A_e$$

$$T_{oe} \overset{(3)}{=} T_o = T_e + \frac{u_e^2}{2c_p} = T_{es} + \frac{u_{es}^2}{2c_p} \Rightarrow u_{es} = \sqrt{2c_p (T_o - T_{es})}$$

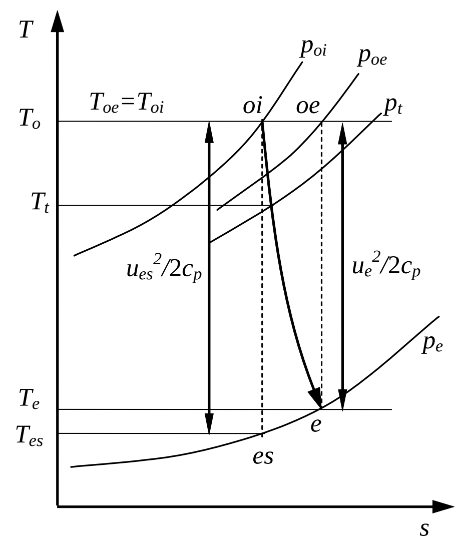
Assumptions:
1. (1) Steady, quasi-1D flow
2. (2) $p_e = p_a$ at design point
3. (3) Adiabatic nozzle
4. (4) Ideal gas behavior, with constant c_p

$$T_{es} = T_o \left(\frac{p_e}{p_o}\right)^{\frac{\gamma-1}{\gamma}} = 1614\ K; \quad c_p \overset{(4)}{=} \frac{\gamma}{\gamma - 1} R_{prop}; \quad R_{prop} \overset{(4)}{=} \frac{\hat{R}}{\hat{M}_{prop}} = \frac{8314\ J/kmol-K}{21.9\ kg/kmol} = 380\ J/kg\text{-}K;$$

$\Rightarrow c_p$ = 2030 *J/kg-K*

$$u_e = u_{es}\sqrt{\eta_n} = \sqrt{2\eta_n c_p (T_o - T_{es})} = 2.20\ km/s$$

$$\dot{m} = \frac{\Im}{u_e} = \frac{1500N}{2.20\times 10^3 m/s} = 0.681\ kg/s \Rightarrow \boxed{\dot{m} = 0.68\ kg/s}$$



$$\dot{m} \overset{(1)}{=} \rho_t A_t u_t = \rho_e A_e u_e$$

$$\rho_e \overset{(4)}{=} \frac{p_e}{R_{prop} T_e} \overset{(2)}{=} \frac{p_a}{R_{prop} T_e}; \quad T_e = T_{oe} - \frac{u_e^2}{2c_p} = 1664\ K \text{ (Note: } T_e > T_{es})$$

$$p_e \overset{(2)}{=} p_a = 100\ kPa \Rightarrow \rho_e = 0.160\ kg/m^3,$$

and $\boxed{A_e = 1.93\times 10^{-3}\ m^2 = 19.3\ cm^2}$

To find $A_t = \frac{\dot{m}}{\rho_t u_t}$; $u_t = \cancel{M_t}_1 \sqrt{\gamma R T_t}$;

$$T_t = \frac{T_o}{1 + \frac{\gamma - 1}{2} \cancel{M_t^2}_1} = \frac{2}{\gamma + 1} T_o = 2565\ K; \quad u_t = 1.09\ km/s$$

$\rho_t \overset{(4)}{=} \frac{p_t}{R_{prop} T_t}$; need p_t. Assume (5) as a reasonable approximation.	**Assumptions (cont'd):** (5) Constant polytropic efficiency from inlet to exit

$$\frac{p_t}{p_o} \overset{(5)}{=} \left(\frac{T_t}{T_o}\right)^{\frac{\gamma}{\eta_{pe}(\gamma-1)}} = \left(\frac{T_t}{T_o}\right)^{\mu}$$

. Can find η_{pe} (or equivalently μ) from inlet and exit conditions because:

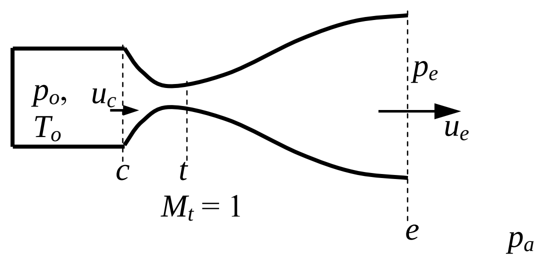
$$\frac{p_o}{p_e} = \left(\frac{T_o}{T_e}\right)^{\mu} \Rightarrow \mu = \frac{\ln\left(p_o / p_e\right)}{\ln\left(T_o / T_e\right)} = 5.648$$ (Correspondingly $\eta_{pe} = 0.947$).

Then $p_t = p_o \left(\frac{T_t}{T_o}\right)^{\mu} = 1.16\ MPa$; $\rho_t \overset{(4)}{=} \frac{p_t}{R_{prop} T_t} = 1.19\ kg/m^3$;

Finally, $$\boxed{A_t = \frac{\dot{m}}{\rho_t u_t} = 5.21 \times 10^{-4}\ m^2 = 5.21\ cm^2}$$

Need: (a) $\dot{m}$ (b) $\Im_{vac}$ (c) M_c

Given: H_2-O_2 propellant, $\hat{M}_{prop}$ = 11.58 *kg/kmol*,
$\gamma = 1.20$, $p_{ot} = 8.26$ *MPa*, $T_t = 3305$ *K*,
$A_t = 750.4\ cm^2$, $p_e = 18.1$ *kPa*, $A_e/A_t = 39.8$,
$A_c/A_t = 3.24$; Flow $c \rightarrow t$ is isentropic



Solution:

Assumptions:
(1) Steady, quasi-1D flow
(2) Supersonic flow in nozzle
(3) Ideal gas behavior, with constant c_p
(4) Adiabatic nozzle

$$\dot{m} \overset{(1)}{=} \rho_t A_t u_t = \rho_e A_e u_e$$

$$u_t = \cancel{M_t}_{1}\sqrt{\gamma R T_t}\ ;$$

$$R_{prop} \overset{(4)}{=} \frac{\hat{R}}{\hat{M}_{prop}} = \frac{8314\ J/kmol-K}{11.58\ kg/kmol} = 718\ J/kg\text{-}K \Rightarrow u_t = 1.69\ km/s$$

$$p_t = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} p_{ot} = 4.66\ MPa;\ \rho_t \overset{(3)}{=} \frac{p_t}{R_{prop}T_t} = 1.96\ kg/m^3\ ;\ \dot{m} = \rho_t A_t u_t \Rightarrow \boxed{\dot{m} = 249\ kg/s}$$

$$\Im_{vac} \overset{(1)}{=} \dot{m}u_e + (p_e - p_a)A_e = \dot{m}u_e + p_e A_e\ \ (p_a = 0 \text{ in vacuum})$$

Know $\dot{m}$, p_e, $A_e = 39.8\ A_t$; need u_e.

$$\dot{m} = \rho_e A_e u_e = \frac{p_e}{R_{prop}T_e}A_e u_e\text{; need } T_e\text{; But } T_{oe} = T_e + \frac{u_e^2}{2c_p} \overset{(4)}{=} T_{ot}\ ;\ c_p \overset{(4)}{=} \frac{\gamma}{\gamma-1}R_{prop} = 4308\ J/kg\text{-}K$$

$$T_{ot} = \frac{2}{\gamma+1}T_t = 3636\ K\ (\text{ or } T_{ot} = T_t + \frac{u_t^2}{2c_p} = 3636\ K)$$

We have 2 equations, mass and energy (i.e. T_o), in 2 unknowns u_e, T_e. Can solve in principle.
Explicitly, from mass conservation we have $T_e = \frac{p_e}{R_{prop}\dot{m}}A_e u_e$. Substituting in equation for T_{ot} we get a quadratic equation in u_e:

$$\frac{u_e^2}{2c_p} + \frac{p_e A_e}{\dot{m}R_{prop}}u_e - T_{ot} = 0;\ u_e^2 + bu_e + c = 0;\ b \equiv \frac{2\gamma}{\gamma-1}\frac{p_e A_e}{\dot{m}};\ c \equiv -2c_p T_{ot}\ ;$$

Substituting numerical values $b = 2.61\times10^3\ m/s$; $c = -3.13\times10^7\ m^2/s^2$ and picking positive root of the quadratic equation $u_e = \frac{-b+\sqrt{b^2-4c}}{2} = 4.44\ km/s$.

Finally $\Im_{vac} = \dot{m}u_e + p_e A_e = 1.15\times10^6\ N$ or $\boxed{\Im_{vac} = 1.15\ MN}$

(Even in a vacuum the pressure term only contributes ~5% because of the large expansion ratio in the diverging section ~40, and hence relatively low pressure in the exit plane.)

We can use relation between area and Mach number because the flow $c \rightarrow t$ is specified as isentropic. Using the spreadsheet for isentropic flow from ERes with $\gamma = 1.20$, and choosing the subsonic solution, $A_c/A_t = 3.24 \Rightarrow \boxed{M_c \approx 0.19}$.